ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$9-\frac{2r}{R}\leq\prod_{cyc}\frac{r_b+r_c}{r_a}\leq\frac{4R}{r}$$

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$$\frac{r_b+r_c}{r_a}=\frac{p\left(\tan\left(\frac{B}{2}\right)+\tan\left(\frac{C}{2}\right)\right)}{p\tan\left(\frac{A}{2}\right)}=\frac{\frac{\sin\left(\frac{B}{2}+\frac{C}{2}\right)}{\cos\left(\frac{B}{2}\right).\cos\left(\frac{C}{2}\right)}}{\tan\left(\frac{A}{2}\right)}=\frac{\cos\left(\frac{A}{2}\right)}{\tan\left(\frac{A}{2}\right).\cos\left(\frac{B}{2}\right).\cos\left(\frac{C}{2}\right)}$$

$$\prod_{cyc}\frac{r_b+r_c}{r_a}=\frac{\cos\left(\frac{A}{2}\right)}{\tan\left(\frac{A}{2}\right).\cos\left(\frac{B}{2}\right).\cos\left(\frac{C}{2}\right)}.\frac{\cos\left(\frac{B}{2}\right)}{\tan\left(\frac{B}{2}\right).\cos\left(\frac{A}{2}\right).\cos\left(\frac{C}{2}\right)}.$$

$$\frac{\cos\left(\frac{C}{2}\right)}{\tan\left(\frac{C}{2}\right).\cos\left(\frac{B}{2}\right).\cos\left(\frac{A}{2}\right)}=\frac{1}{\tan\left(\frac{A}{2}\right).\tan\left(\frac{B}{2}\right).\tan\left(\frac{C}{2}\right).\cos\left(\frac{A}{2}\right).\cos\left(\frac{B}{2}\right).\cos\left(\frac{C}{2}\right)}=$$

$$=\frac{1}{\prod_{cyc}\sin\left(\frac{A}{2}\right)}=\frac{4R}{r}$$

$$\frac{r_b+r_c}{r_a}=\frac{\frac{F}{p-b}+\frac{F}{p-c}}{\frac{F}{p-a}}=\frac{a(p-a)}{(p-b)(p-c)}$$

$$\prod_{cyc}\frac{r_b+r_c}{r_a}=\frac{a(p-a)}{(p-b)(p-c)}.\frac{b(p-b)}{(p-a)(p-c)}.\frac{c(p-c)}{(p-a)(p-b)}=$$

$$=\frac{abc}{(p-a)(p-b)(p-c)}=\frac{4RF.p}{p(p-a)(p-b)(p-c)}=$$

$$\frac{4RF.p}{F^2}=\frac{4Rp}{F}=\frac{4R.p}{pr}=\frac{4R}{r}$$

$$\frac{4R}{r}\geq 9-\frac{2r}{R};\ \ Let\ \frac{R}{r}=x,\ \ Then:\ \ 4x\geq 9-\frac{2}{x}\ \ \rightarrow$$

$$4x^2 - 9x + 2 \geq 0 \rightarrow (4x-1)(x-2) \geq 0 \quad x \rightarrow \quad x \geq 2$$

$$\frac{R}{r} \geq 2 \quad (True)$$

$$Equality \ holds \ if \ a = b = c$$