ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\left(\sum_{cyc}\frac{1}{h_a^3}+\frac{R}{2r^2s^2}\right)\left(\sum_{cyc}\frac{1}{h_bh_c}\right)\geq\frac{4}{81r^5}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc}\frac{1}{h_a^3}+\frac{R}{2r^2s^2}\right)\left(\sum_{cyc}\frac{1}{h_bh_c}\right)=$$

$$=\left(\frac{2s(s^2-6Rr-3r^2)}{8r^3s^3}+\frac{R}{2r^2s^2}\right)\left(\frac{s^2+4Rr+r^2}{2R.\frac{2r^2s^2}{R}}\right)=$$

$$=\frac{(s^2-4Rr-3r^2)(s^2+4Rr+r^2)}{16r^5s^4}\overset{?}{\geq}\frac{4}{81r^5}$$

$$\Leftrightarrow 17s^4-162r^2s^2-r^2(1296R^2+1296Rr+243r^2)\underset{①}{\overset{?}{\geq}}0$$

$$\text{Now, via Gerretsen, LHS of ①}\geq$$

$$17s^2(16Rr-5r^2)-162r^2s^2-r^2(1296R^2+1296Rr+243r^2)\overset{?}{\geq}0$$

$$\Leftrightarrow (272R-247r)s^2\underset{②}{\overset{?}{\geq}}r(1296R^2+1296Rr+243r^2)\ \textit{and indeed,}$$

$$(272R-247r)s^2\overset{\text{Gerretsen}}{\geq}(272R-247r)(16Rr-5r^2)\overset{?}{\geq}$$

$$r(1296R^2+1296Rr+243r^2)\Leftrightarrow 16r(R-2r)(191R-31r)\overset{?}{\geq}0$$

$$\rightarrow\text{true}\because R\overset{\text{Euler}}{\geq}2r\Rightarrow ②\Rightarrow ①\text{ is true and so,}$$

$$\left(\sum_{cyc}\frac{1}{h_a^3}+\frac{R}{2r^2s^2}\right)\left(\sum_{cyc}\frac{1}{h_bh_c}\right)\geq\frac{4}{81r^5}\ \forall\,\Delta\,ABC,$$

$$''=''\ \text{iff }\Delta\,ABC\text{ is equilateral (QED)}$$