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In ΔABC the following relationship holds:

$$\sum_{cyc} \sqrt[3]{\left(1+\frac{6r}{h_a}\right)\left(\frac{r}{h_b}\right)} \geq 3$$

Proposed by Marin Chirciu-Romania

Solution by Jenish Rijal-Nepal

By applying Hölder's *Inequality, we obtain*:

$$\sum_{cyc} \sqrt[3]{\left(1+\frac{6r}{h_a}\right)\left(\frac{r}{h_b}\right)} = \sum_{cyc}\left(\sqrt[3]{\left(1+\frac{6r}{h_a}\right)}\sqrt[3]{\left(\frac{r}{h_b}\right)}\cdot\sqrt[3]{1}\right) \leq \sqrt[3]{\sum_{cyc}\left(1+\frac{6r}{h_a}\right)\cdot\sum_{cyc}\left(\frac{r}{h_b}\right)\cdot\sum_{cyc}(1)}$$

Now,

$$\sqrt[3]{\sum_{cyc}\left(1+\frac{6r}{h_a}\right)\cdot\sum_{cyc}\left(\frac{r}{h_b}\right)\cdot\sum_{cyc}(1)} = \sqrt[3]{\left(3+\sum_{cyc}\frac{6r}{h_a}\right)\cdot\left(r\cdot\sum_{cyc}\frac{1}{h_b}\right)\cdot 3} =$$

$$= \sqrt[3]{\left(3+6r\cdot\sum_{cyc}\frac{1}{h_a}\right)\left(r\cdot\sum_{cyc}\frac{1}{h_b}\right)\cdot 3} = \sqrt[3]{\left(3+6r\cdot\frac{1}{r}\right)\left(r\cdot\frac{1}{r}\right)\cdot 3} = \sqrt[3]{27} = 3$$

$$\therefore \sum_{cyc} \sqrt[3]{\left(1+\frac{6r}{h_a}\right)\left(\frac{r}{h_b}\right)} \leq 3$$

Equality holds iff ΔABC is equilateral.