ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{h_a}{h_a^2+r^2}\le\frac{9}{10r}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{h_a}{h_a^2+r^2}\overset{?}{\le}\frac{9}{10r}\Leftrightarrow\sum_{cyc}\frac{\frac{h_a}{r}}{\frac{h_a^2}{r^2}+1}\overset{?}{\le}\frac{9}{10}\Leftrightarrow\sum_{cyc}\frac{\frac{3}{x}}{\frac{9}{x^2}+1}\overset{?}{\le}\frac{9}{10}$$

$$\left(x=\frac{3r}{h_a},y=\frac{3r}{h_b},y=\frac{3r}{h_c}\right)\Leftrightarrow\sum_{cyc}\frac{x}{9+x^2}\underset{(*)}{\overset{?}{\le}}\frac{3}{10}$$

Now, $f(t)=\frac{t}{9+t^2}\ \forall\, t\in(0,3)$ is cocave as $f''(t)=\frac{2t(t^2-27)}{(t^2+9)^3}<0$

$(\because t^2<9<27)\ \therefore$ as $\sum_{cyc}x=3\Rightarrow 0<x,y,z<3$, hence : $\sum_{cyc}\frac{x}{9+x^2}$

$$\overset{\text{Jensen}}{\le}3.\frac{\frac{\sum_{cyc}x}{3}}{9+\left(\frac{\sum_{cyc}x}{3}\right)^2}=\frac{3}{9+1}=\frac{3}{10}\Rightarrow(*)\text{ is true and so,}$$

$$\sum_{cyc}\frac{h_a}{h_a^2+r^2}\le\frac{9}{10r}\ \forall\,\Delta\,\text{ABC},''=''\ \text{iff }\Delta\,\text{ABC is equilateral (QED)}$$