ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{1}{3r^2} \le \sum_{cyc} \frac{1}{r_a^2} \le \frac{R^2}{12r^4}$$

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$$\sum_{cyc} \frac{1}{r_a^2} = \sum_{cyc} \frac{(s-a)^2}{r^2s^2} = \frac{3s^2 - 2s.2s + 2(s^2 - 4Rr - r^2)}{r^2s^2} = \frac{s^2 - 8Rr - 2r^2}{r^2s^2}$$

$$\overset{?}{\le} \frac{R^2}{12r^4} \Leftrightarrow (R^2 - 12r^2)s^2 + 24r^3(4R + r) \underbrace{\overset{?}{\ge}}_{(*)} 0 \text{ and it's trivially true if :}$$

$$R^2 - 12r^2 \ge 0 \text{ and when : } R^2 - 12r^2 < 0, \text{then : LHS of } (*) \overset{\text{Gerretsen}}{\ge}$$

$$(R^2 - 12r^2)(4R^2 + 4Rr + 3r^2) + 24r^3(4R + r) \overset{?}{\ge} 0$$

$$\Leftrightarrow 4t^4 + 4t^3 - 45t^2 + 48t - 12 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)(4t^3 + 12t^2 - 21t + 6) \overset{?}{\ge} 0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (*) \text{ is true } \forall \Delta \text{ ABC} \therefore \sum_{cyc} \frac{1}{r_a^2} \le \frac{R^2}{12r^4} \text{ and again,}$$

$$\sum_{cyc} \frac{1}{r_a^2} = \frac{s^2 - 8Rr - 2r^2}{r^2s^2} \overset{?}{\ge} \frac{1}{3r^2} \Leftrightarrow s^2 \overset{?}{\ge} 12Rr + 3r^2 \to \text{true}$$

$$\because s^2 \overset{\text{Gerretsen}}{\ge} 12Rr + 3r^2 + 4r(R - 2r) \overset{\text{Euler}}{\ge} 12Rr + 3r^2 \therefore \sum_{cyc} \frac{1}{r_a^2} \ge \frac{1}{3r^2} \text{ and so,}$$

$$\frac{1}{3r^2} \le \sum_{cyc} \frac{1}{r_a^2} \le \frac{R^2}{12r^4} \ \forall \Delta \text{ ABC}, '' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$