ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \frac{1}{h_a^2} \leq \sum \frac{1}{r_a^2}$$

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Solution by Tapas Das-India

$$\sum \frac{1}{h_a^2} = \frac{1}{4F^2}\sum a^2 = \frac{s^2 - r^2 - 4Rr}{2s^2r^2}$$

$$\sum \frac{1}{r_a^2} = \left(\sum \frac{1}{r_a}\right)^2 - 2\sum \frac{1}{r_a r_b} = \frac{1}{r^2} - \frac{2(4R+r)}{s^2 r} = \frac{s^2 - 2r(4R+r)}{s^2 r^2}$$

$$We\ need\ to\ show:$$

$$\frac{s^2 - 2r(4R+r)}{s^2r^2} \geq \frac{s^2 - r^2 - 4Rr}{2s^2r^2}$$

$$2s^2 - 4r(4R+r) \geq s^2 - r^2 - 4Rr$$

$$s^2 \geq 4r(4R+r) - r^2 - 4Rr$$

$$s^2 \geq 16Rr + 4r^2 - r^2 - 4Rr$$

$$s^2 \geq 12Rr + 3r^2$$

$$s^2 \overset{GERRETSEN}{\geq} 16Rr - 5r^2 \geq 12Rr + 3r^2$$

$$4Rr \geq 8r^2$$

$$R \overset{EULER}{\geq} 2r$$

Equality holds for an equilateral triangle.