ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{\cot^2 A\cot^2 B}{\cot C\,(\cot A+\cot B)}\ge\frac{1}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{\tan^2\frac{B}{2}\tan^2\frac{C}{2}}{\tan\frac{A}{2}.\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)}=\frac{1}{s^2}.\sum_{cyc}\frac{r_b^2r_c^2}{r_a(r_b+r_c)}\overset{\text{Bergstrom}}{\ge}\frac{1}{s^2}.\frac{\left(\sum_{cyc}r_ar_b\right)^2}{2\sum_{cyc}r_ar_b}$$

$$=\frac{s^2}{2s^2}=\frac{1}{2}\therefore\sum_{cyc}\frac{\tan^2\frac{B}{2}\tan^2\frac{C}{2}}{\tan\frac{A}{2}.\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)}\ge\frac{1}{2}\to (1)$$

Let us consider a Δ A′B′C′ with angles $A'\equiv(\pi-2A), B'\equiv(\pi-2B)$ and $C'\equiv(\pi-2C)$ & then : $\cos A'\cos B'\cos C'=\cos(\pi-2A)\cos(\pi-2B)\cos(\pi-2C)$
$=-\cos 2A\cos 2B\cos 2C=1+4\cos A\cos B\cos C>0$
($\because$ Δ ABC being acute $\Rightarrow\cos A\cos B\cos C>0$) $\Rightarrow$ Δ A′B′C′ is acute and hence,

implementing (1) on Δ A′B′C′, we get : $$\sum_{cyc}\frac{\tan^2\frac{\pi-2B}{2}\tan^2\frac{\pi-2C}{2}}{\tan\frac{\pi-2A}{2}.\left(\tan\frac{\pi-2B}{2}+\tan\frac{\pi-2C}{2}\right)}\ge\frac{1}{2}$$

$$\Rightarrow\sum_{cyc}\frac{\cot^2 B\cot^2 C}{\cot A\,(\cot B+\cot C)}\ge\frac{1}{2}\ \forall\text{ acute }\Delta\text{ ABC},$$

" = " iff Δ ABC is equilateral (QED)