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In ΔABC the following relationship holds:

$$\frac{9r^2}{2R^2} \le \sum_{cyc} \frac{1-\cos(A)}{1+tan^2\left(\frac{A}{2}\right)} \le 1+\frac{r^2}{2R^2}$$

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Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{1-\cos(A)}{1+tan^2\left(\frac{A}{2}\right)} = \frac{2sin^2\left(\frac{A}{2}\right)}{\frac{1}{cos^2\left(\frac{A}{2}\right)}} = 2\left(sin^2\left(\frac{A}{2}\right).cos^2\left(\frac{A}{2}\right)\right) = \frac{1}{2}(\sin(A))^2 \ *$$

$$\sum_{cyc} \frac{s^2-r^2-4Rr}{2R^2} \overset{Gerretsen}{\overbrace{\le}}$$

$$\le \frac{4R^2+4Rr+3r^2-r^2-4Rr}{2R^2} = \frac{4R^2+2r^2}{2R^2} = 2+\frac{r^2}{R^2} \ (1)$$

$$\sum_{cyc} sin^2(A) = \frac{s^2-r^2-4Rr}{2R^2} \ge \frac{16Rr-5r^2-r^2-4Rr}{2R^2} =$$

$$= \frac{12Rr-2r^2}{2R^2} \overset{Euler}{\overbrace{\ge}} \frac{18r^2}{2R^2} = \frac{9r^2}{R^2} \quad (2)$$

$From\ (*)\,,(1)\ and\ (2)\ we\ get\ that:$

$$\frac{9r^2}{2R^2} \le \sum_{cyc} \frac{1-\cos(A)}{1+tan^2\left(\frac{A}{2}\right)} \le 1+\frac{r^2}{2R^2}$$

$Equality\ holds\ for\ a=b=c.$