ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$-1-\frac{2r^2}{R^2}\le\sum_{cyc}\frac{1-\tan^2 A}{1+\tan^2 A}\le 3-\frac{18r^2}{R^2}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1-\tan^2 A}{1+\tan^2 A}=\sum_{cyc}\frac{\cos^2 A-\sin^2 A}{\cos^2 A+\sin^2 A}=\sum_{cyc}\cos 2A$$

$$=-1-4\cos A\cos B\cos C=-1-\frac{s^2-4R^2-4Rr-r^2}{R^2}$$

$$\therefore\sum_{cyc}\frac{1-\tan^2 A}{1+\tan^2 A}=\frac{3R^2+4Rr+r^2-s^2}{R^2}\text{ and }\because s^2\overset{\text{Gerretsen}}{\ge}16Rr-5r^2\text{ and}$$

$$s^2\overset{\text{Gerretsen}}{\le}4R^2+4Rr+3r^2\ \therefore\sum_{cyc}\frac{1-\tan^2 A}{1+\tan^2 A}\le\frac{3R^2+4Rr+r^2-16Rr+5r^2}{R^2}$$

$$=3-\frac{12Rr-6r^2}{R^2}\overset{\text{Euler}}{\le}3-\frac{24r^2-6r^2}{R^2}=3-\frac{18r^2}{R^2}\text{ and}$$

$$\sum_{cyc}\frac{1-\tan^2 A}{1+\tan^2 A}\ge\frac{3R^2+4Rr+r^2-(4R^2+4Rr+3r^2)}{R^2}=-1-\frac{2r^2}{R^2}\text{ and so,}$$

$$-1-\frac{2r^2}{R^2}\le\sum_{cyc}\frac{1-\tan^2 A}{1+\tan^2 A}\le 3-\frac{18r^2}{R^2}\ \forall\,\Delta\text{ ABC},$$

$'' = ''$ iff Δ ABC is equilateral (QED)