ROMANIAN MATHEMATICAL MAGAZINE

In any $\Delta$ ABC the following relationship holds :

$$8 \le \prod_{cyc} \frac{m_b + m_c}{m_a} \le \frac{8}{729}\left(\frac{4R}{r} + 1\right)^3$$

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$$\prod_{cyc} \frac{m_b + m_c}{m_a} = \frac{\left(\sum_{cyc} m_a\right)\left(\sum_{cyc} m_a m_b\right) - m_a m_b m_c}{m_a m_b m_c}$$

$$= \left(\sum_{cyc} m_a\right)\left(\sum_{cyc} \frac{1}{m_a}\right) - 1 \le \left(\sum_{cyc} m_a\right)\left(\sum_{cyc} \frac{1}{h_a}\right) - 1 \overset{Bager}{\le} \frac{4R + r}{r} - 1$$

$$\therefore \prod_{cyc} \frac{m_b + m_c}{m_a} \le \frac{4R}{r} \rightarrow (1) \text{ and again, } \frac{8}{729}\left(\frac{4R}{r} + 1\right)^3 \overset{Euler}{\ge} \frac{8}{81}\left(\frac{4R}{r} + 1\right)^2 \overset{?}{\ge} \frac{4R}{r}$$

$$\Leftrightarrow 2(4R + r)^2 \overset{?}{\ge} 81Rr \Leftrightarrow 32R^2 - 65Rr + 2r^2 \overset{?}{\ge} 0 \Leftrightarrow (32R - r)(R - 2r) \overset{?}{\ge} 0$$

$$\rightarrow \text{true via Euler} \therefore \frac{8}{729}\left(\frac{4R}{r} + 1\right)^3 \ge \frac{4R}{r} \overset{via (1)}{\ge} \prod_{cyc} \frac{m_b + m_c}{m_a} \text{ and since}$$

$$\prod_{cyc} \frac{m_b + m_c}{m_a} \overset{Cesaro}{\ge} 8 \text{ and so, } 8 \le \prod_{cyc} \frac{m_b + m_c}{m_a} \le \frac{8}{729}\left(\frac{4R}{r} + 1\right)^3 \ \forall \Delta \text{ ABC},$$

$'' ='' $ iff $\Delta$ ABC is equilateral (QED)