ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$8 \leq \prod_{cyc} \frac{m_b + m_c}{m_a} \leq \left(\frac{R}{r}\right)^3$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sqrt[3]{\prod_{cyc} \frac{m_b + m_c}{m_a}} \overset{AM-GM}{\leq} \frac{2}{3}\left(\sum_{cyc} m_a\right) \cdot \frac{1}{\sqrt[3]{m_a m_b m_c}} \overset{Bager}{\leq} \frac{2}{3} \cdot \frac{4R + r}{\sqrt[3]{h_a h_b h_c}} \overset{Euler}{\leq}$$

$$\frac{2}{3} \cdot \frac{\frac{9R}{2}}{\sqrt[3]{\frac{2r^2 s^2}{R}}} \overset{Gerretsen + Euler}{\leq} \frac{2}{3} \cdot \frac{\frac{9R}{2}}{\sqrt[3]{\frac{r^2 . 27Rr}{R}}} = \frac{R}{r} \therefore \prod_{cyc} \frac{m_b + m_c}{m_a} \leq \left(\frac{R}{r}\right)^3 \text{ and}$$

$$\text{via Cesaro, } \prod_{cyc} \frac{m_b + m_c}{m_a} \geq 8 \text{ and so, } 8 \leq \prod_{cyc} \frac{m_b + m_c}{m_a} \leq \left(\frac{R}{r}\right)^3 \quad \forall \Delta ABC,$$

$$'' = '' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$