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In any ΔABC the following relationship holds :

$$\sum_{\text{cyc}} \frac{1}{r_a \cdot \sqrt{\frac{1}{r_b} + \frac{1}{r_c}}} \geq \sqrt{\frac{3}{2r}}$$

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$$\text{Let } \frac{3r}{r_a} = x, \frac{3r}{r_b} = y, \frac{3r}{r_c} = z \text{ and then : } \sum_{\text{cyc}} x = 3 \rightarrow \text{(i)}$$

$$\begin{aligned} \text{Now, } \sum_{\text{cyc}} \frac{1}{r_a \cdot \sqrt{\frac{1}{r_b} + \frac{1}{r_c}}} &= \frac{1}{\sqrt{r}} \cdot \sum_{\text{cyc}} \frac{\frac{r}{r_a}}{\sqrt{\frac{r}{r_b} + \frac{r}{r_c}}} = \frac{1}{\sqrt{r}} \cdot \sum_{\text{cyc}} \frac{\frac{x}{3}}{\sqrt{\frac{y+z}{3}}} = \frac{1}{\sqrt{3r}} \cdot \sum_{\text{cyc}} \frac{x^2}{x \cdot \sqrt{y+z}} \\ &\stackrel{\text{Bergstrom}}{\geq} \frac{1}{\sqrt{3r}} \cdot \frac{(\sum_{\text{cyc}} x)^2}{\sum_{\text{cyc}} (x \cdot \sqrt{y+z})} = \frac{1}{\sqrt{3r}} \cdot \frac{(\sum_{\text{cyc}} x)^2}{\sum_{\text{cyc}} (\sqrt{x} \cdot \sqrt{xy+zx})} \stackrel{\text{CBS}}{\geq} \\ &\frac{1}{\sqrt{3r}} \cdot \frac{(\sum_{\text{cyc}} x)^2}{\sqrt{2(\sum_{\text{cyc}} x)(\sum_{\text{cyc}} xy)}} \geq \frac{1}{\sqrt{3r}} \cdot \frac{(\sum_{\text{cyc}} x)^2}{\sqrt{2(\sum_{\text{cyc}} x) \cdot \frac{1}{3} (\sum_{\text{cyc}} x)^2}} = \frac{\sqrt{\sum_{\text{cyc}} x} \text{ via (i)}}{\sqrt{2r}} = \frac{\sqrt{3}}{\sqrt{2r}} \end{aligned}$$

and so, $\sum_{\text{cyc}} \frac{1}{r_a \cdot \sqrt{\frac{1}{r_b} + \frac{1}{r_c}}} \geq \sqrt{\frac{3}{2r}} \forall \Delta ABC, "=" \text{ iff } \Delta ABC \text{ is equilateral}$