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In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{1}{h_a\sqrt{\frac{1}{h_b}+\frac{1}{h_c}}}\ge\sqrt{\frac{3}{2r}}$$

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$$\text{Let } \frac{3r}{h_a}=x,\frac{3r}{h_b}=y,\frac{3r}{h_c}=z \text{ and then : } \sum_{cyc}x=3\to(\text{i})$$

$$\text{Now, }\sum_{cyc}\frac{1}{h_a.\sqrt{\frac{1}{h_b}+\frac{1}{h_c}}}=\frac{1}{\sqrt{r}}.\sum_{cyc}\frac{\frac{r}{h_a}}{\sqrt{\frac{r}{h_b}+\frac{r}{h_c}}}=\frac{1}{\sqrt{r}}.\sum_{cyc}\frac{\frac{x}{3}}{\sqrt{\frac{y+z}{3}}}=\frac{1}{\sqrt{3r}}.\sum_{cyc}\frac{x^2}{x.\sqrt{y+z}}$$

$$\overset{\text{Bergstrom}}{\ge}\frac{1}{\sqrt{3r}}.\frac{\left(\sum_{cyc}x\right)^2}{\sum_{cyc}\left(x.\sqrt{y+z}\right)}=\frac{1}{\sqrt{3r}}.\frac{\left(\sum_{cyc}x\right)^2}{\sum_{cyc}\left(\sqrt{x}.\sqrt{xy+zx}\right)}\overset{\text{CBS}}{\ge}$$

$$\frac{1}{\sqrt{3r}}.\frac{\left(\sum_{cyc}x\right)^2}{\sqrt{2\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)}}\ge\frac{1}{\sqrt{3r}}.\frac{\left(\sum_{cyc}x\right)^2}{\sqrt{2\left(\sum_{cyc}x\right).\frac{1}{3}\left(\sum_{cyc}x\right)^2}}=\frac{\sqrt{\sum_{cyc}x}}{\sqrt{2r}}\overset{\text{via (i)}}{=}\frac{\sqrt{3}}{\sqrt{2r}}$$

$$\text{and so, }\sum_{cyc}\frac{1}{h_a.\sqrt{\frac{1}{h_b}+\frac{1}{h_c}}}\ge\sqrt{\frac{3}{2r}}\ \forall\,\Delta\,\text{ABC},''=''\text{ iff }\Delta\,\text{ABC is equilateral (QED)}$$