

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$5 \sum_{\text{cyc}} \frac{r^2}{r_a^2} \leq 1 + 6 \sum_{\text{cyc}} \frac{r^3}{r_a^3}$$

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$$\begin{aligned} & \text{Let } \frac{3r}{r_a} = x, \frac{3r}{r_b} = y, \frac{3r}{r_c} = z \text{ and then : } \sum_{\text{cyc}} x = 3 \rightarrow \text{(i)} \\ \text{Now, } & 5 \sum_{\text{cyc}} \frac{r^2}{r_a^2} \stackrel{?}{\leq} 1 + 6 \sum_{\text{cyc}} \frac{r^3}{r_a^3} \Leftrightarrow 5 \sum_{\text{cyc}} \frac{x^2}{9} \stackrel{?}{\leq} 1 + 6 \sum_{\text{cyc}} \frac{x^3}{27} \Leftrightarrow 2 \sum_{\text{cyc}} x^3 + 9 \stackrel{?}{\geq} 5 \sum_{\text{cyc}} x^2 \\ & \Leftrightarrow 2 \sum_{\text{cyc}} x^3 + \frac{9}{27} \left(\sum_{\text{cyc}} x \right)^3 \stackrel{?}{\geq} \frac{5}{3} \left(\sum_{\text{cyc}} x^2 \right) \left(\sum_{\text{cyc}} x \right) \left(\because \sum_{\text{cyc}} x = 3 \text{ via (i)} \right) \\ & \Leftrightarrow 6 \sum_{\text{cyc}} x^3 + \left(\sum_{\text{cyc}} x \right)^3 \stackrel{?}{\geq} 5 \left(\sum_{\text{cyc}} x^2 \right) \left(\sum_{\text{cyc}} x \right) \\ & \Leftrightarrow \sum_{\text{cyc}} x^3 + 3xyz \stackrel{?}{\geq} \sum_{\text{cyc}} x^2y + \sum_{\text{cyc}} xy^2 \rightarrow \text{true via Schur} \\ \therefore & 5 \sum_{\text{cyc}} \frac{r^2}{r_a^2} \leq 1 + 6 \sum_{\text{cyc}} \frac{r^3}{r_a^3} \forall \Delta ABC, " = " \text{ iff } \Delta ABC \text{ is equilateral (QED)} \end{aligned}$$