ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$5\sum_{cyc}\frac{r^2}{h_a^2} \le 1 + 6\sum_{cyc}\frac{r^3}{h_a^3}$$

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$$\text{Let } \frac{3r}{h_a} = x, \frac{3r}{h_b} = y, \frac{3r}{h_c} = z \text{ and then : } \sum_{cyc} x = 3 \to (i)$$

$$\text{Now, } 5\sum_{cyc}\frac{r^2}{h_a^2} \le 1 + 6\sum_{cyc}\frac{r^3}{h_a^3} \Leftrightarrow 5\sum_{cyc}\frac{x^2}{9} \overset{?}{\le} 1 + 6\sum_{cyc}\frac{x^3}{27} \Leftrightarrow 2\sum_{cyc}x^3 + 9 \overset{?}{\ge} 5\sum_{cyc}x^2$$

$$\Leftrightarrow 2\sum_{cyc}x^3 + \frac{9}{27}\left(\sum_{cyc}x\right)^3 \overset{?}{\ge} \frac{5}{3}\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x\right) \left(\because \sum_{cyc}x = 3 \text{ via } (i)\right)$$

$$\Leftrightarrow 6\sum_{cyc}x^3 + \left(\sum_{cyc}x\right)^3 \overset{?}{\ge} 5\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x\right)$$

$$\Leftrightarrow \sum_{cyc}x^3 + 3xyz \overset{?}{\ge} \sum_{cyc}x^2y + \sum_{cyc}xy^2 \to \text{true via Schur}$$

$$\therefore 5\sum_{cyc}\frac{r^2}{h_a^2} \le 1 + 6\sum_{cyc}\frac{r^3}{h_a^3}\ \forall\ \Delta\ ABC, '' ='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$