ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$\sum_{cyc}\frac{\frac{r}{r_a^2}+\frac{1}{r_b}}{\frac{1}{r_b}+\frac{1}{r_c}}\geq 2$$

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$$\text{Let } \frac{3r}{r_a}=x,\frac{3r}{r_b}=y,\frac{3r}{r_c}=z \text{ and then : } \sum_{cyc}x=3\rightarrow(i)$$

$$\text{Now, } \sum_{cyc}\frac{\frac{r}{r_a^2}+\frac{1}{r_b}}{\frac{1}{r_b}+\frac{1}{r_c}}=\sum_{cyc}\frac{\frac{r^2}{r_a^2}+\frac{r}{r_b}}{\frac{r}{r_b}+\frac{r}{r_c}}=\sum_{cyc}\frac{\frac{x^2}{9}+\frac{y}{3}}{\frac{y}{3}+\frac{z}{3}}\overset{?}{\geq}2\Leftrightarrow\sum_{cyc}\frac{x^2}{y+z}+3\sum_{cyc}\frac{y}{y+z}\overset{?}{\geq}6$$

$$\Leftrightarrow\sum_{cyc}\frac{x^2}{y+z}+\sum_{cyc}\frac{y(x+y+z)}{y+z}\overset{?}{\geq}6\left(\because 3=\sum_{cyc}x \text{ via (i)}\right)$$

$$\Leftrightarrow\sum_{cyc}\frac{x^2+xy}{y+z}+\sum_{cyc}y\overset{?}{\geq}6\Leftrightarrow\sum_{cyc}\frac{x^2+xy}{y+z}\overset{?}{\geq}3\left(\because\sum_{cyc}x=3\right)$$

$$\Leftrightarrow\sum_{cyc}\frac{x^2+xy}{y+z}\overset{?}{\geq}\sum_{cyc}x\Leftrightarrow\sum_{cyc}\left(\frac{x^2+xy}{y+z}-x\right)\overset{?}{\geq}0\Leftrightarrow\sum_{cyc}\frac{x^2-xz}{y+z}\overset{?}{\geq}0$$

$$\Leftrightarrow\frac{1}{(x+y)(y+z)(z+x)}\cdot\sum_{cyc}\left((x^2-xz)\left(x^2+\sum_{cyc}xy\right)\right)\overset{?}{\geq}0$$

$$\Leftrightarrow\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^2-\sum_{cyc}xy\right)+\sum_{cyc}x^4-\sum_{cyc}xy^3\boxed{\overset{?}{\geq}}0\ (*)$$

$$\text{Now, } x^4+y^4+y^4+y^4\underbrace{\overset{AM-GM}{\geq}}_{①}4xy^3, y^4+z^4+z^4+z^4\underbrace{\overset{AM-GM}{\geq}}_{②}4yz^3 \text{ and}$$

$$z^4+x^4+x^4+x^4\underbrace{\overset{AM-GM}{\geq}}_{③}4zx^3 \text{ and } ①+②+③\Rightarrow\sum_{cyc}x^4\geq\sum_{cyc}xy^3 \text{ and this}$$

$$\text{combined with } \sum_{cyc}x^2\geq\sum_{cyc}xy\Rightarrow(*) \text{ is true } \therefore \sum_{cyc}\frac{\frac{r}{r_a^2}+\frac{1}{r_b}}{\frac{1}{r_b}+\frac{1}{r_c}}\geq 2\ \forall\ \Delta ABC,$$

$$''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$