ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc} \frac{\frac{r}{h_a^2} + \frac{1}{h_b}}{\frac{1}{h_b} + \frac{1}{h_c}} \geq 2$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } \frac{3r}{h_a} = x, \frac{3r}{h_b} = y, \frac{3r}{h_c} = z \text{ and then : } \sum_{cyc} x = 3 \to (\text{i})$$

$$\text{Now, } \sum_{cyc} \frac{\frac{r}{h_a^2} + \frac{1}{h_b}}{\frac{1}{h_b} + \frac{1}{h_c}} = \sum_{cyc} \frac{\frac{r^2}{h_a^2} + \frac{r}{h_b}}{\frac{r}{h_b} + \frac{r}{h_c}} = \sum_{cyc} \frac{\frac{x^2}{9} + \frac{y}{3}}{\frac{y}{3} + \frac{z}{3}} \overset{?}{\geq} 2 \Leftrightarrow \sum_{cyc} \frac{x^2}{y+z} + 3\sum_{cyc} \frac{y}{y+z} \overset{?}{\geq} 6$$

$$\Leftrightarrow \sum_{cyc} \frac{x^2}{y+z} + \sum_{cyc} \frac{y(x+y+z)}{y+z} \overset{?}{\geq} 6 \left(\because 3 = \sum_{cyc} x \text{ via (i)}\right)$$

$$\Leftrightarrow \sum_{cyc} \frac{x^2 + xy}{y+z} + \sum_{cyc} y \overset{?}{\geq} 6 \Leftrightarrow \sum_{cyc} \frac{x^2 + xy}{y+z} \overset{?}{\geq} 3 \left(\because \sum_{cyc} x = 3\right)$$

$$\Leftrightarrow \sum_{cyc} \frac{x^2 + xy}{y+z} \overset{?}{\geq} \sum_{cyc} x \Leftrightarrow \sum_{cyc} \left(\frac{x^2 + xy}{y+z} - x\right) \overset{?}{\geq} 0 \Leftrightarrow \sum_{cyc} \frac{x^2 - xz}{y+z} \overset{?}{\geq} 0$$

$$\Leftrightarrow \frac{1}{(x+y)(y+z)(z+x)} \cdot \sum_{cyc} \left((x^2 - xz) \left(x^2 + \sum_{cyc} xy \right) \right) \overset{?}{\geq} 0$$

$$\Leftrightarrow \left(\sum_{cyc} xy\right)\left(\sum_{cyc} x^2 - \sum_{cyc} xy\right) + \sum_{cyc} x^4 - \sum_{cyc} xy^3 \underset{(*)}{\overset{?}{\geq}} 0$$

$$\text{Now, } x^4 + y^4 + y^4 + y^4 \underset{①}{\overset{AM-GM}{\geq}} 4xy^3, y^4 + z^4 + z^4 + z^4 \underset{②}{\overset{AM-GM}{\geq}} 4yz^3 \text{ and}$$

$$z^4 + x^4 + x^4 + x^4 \underset{③}{\overset{AM-GM}{\geq}} 4zx^3 \text{ and } ① + ② + ③ \Rightarrow \sum_{cyc} x^4 \geq \sum_{cyc} xy^3 \text{ and this}$$

$$\text{combined with } \sum_{cyc} x^2 \geq \sum_{cyc} xy \Rightarrow (*) \text{ is true } \therefore \sum_{cyc} \frac{\frac{r}{h_a^2} + \frac{1}{h_b}}{\frac{1}{h_b} + \frac{1}{h_c}} \geq 2\ \forall\ \Delta \text{ ABC},$$

$$'' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$