ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$1 \leq \sum \tan^2 \frac{A}{2} \leq \frac{R}{r} - 1$$

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Solution by Tapas Das-India

$$\sum \tan^2 \frac{A}{2} = \left(\frac{4R + r}{s}\right)^2 - 2$$

$$We\ need\ to\ show:$$

$$\left(\frac{4R + r}{s}\right)^2 - 2 \leq \frac{R}{r} - 1 \ or \ \left(\frac{4R + r}{s}\right)^2 \leq \frac{R}{r} + 1$$

$$(4R + r)^2 \leq s^2 \left(\frac{R}{r} + 1\right) \ or \ (4R + r)^2 \overset{Gerretsen}{\leq} (16Rr - 5r^2)\left(\frac{R + r}{r}\right)$$

$$16R^2 + 8Rr + r^2 \leq 16R^2 + 11Rr - 5r^2 \ or \ 3Rr \geq 6r^2 \ or \ R \geq 2r \ true\ by\ Euler$$

$$\sum \tan^2 \frac{A}{2} = \left(\frac{4R + r}{s}\right)^2 - 2 \overset{Douc\grave{e}t}{\geq} 3 - 2 = 1$$

Equality holds for an equilateral triangle.