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In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{r_b\cdot\sqrt{r_c}}{r_a^2}\ge\sqrt{\frac{3}{r}}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{r_b\cdot\sqrt{r_c}}{r_a^2}=\sum_{cyc}\frac{\left(\frac{1}{r_a}\right)^2}{\frac{1}{r_b}\cdot\sqrt{\frac{1}{r_c}}}\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}\frac{1}{r_a}\right)^2}{\sum_{cyc}\left(\frac{1}{r_b}\cdot\sqrt{\frac{1}{r_c}}\right)}\overset{\text{CBS}}{\ge}\frac{\frac{1}{r^2}}{\sqrt{\sum_{cyc}\frac{1}{r_br_c}}\cdot\sqrt{\sum_{cyc}\frac{1}{r_b}}}$$

$$=\frac{\sqrt{r}}{r^2\cdot\sqrt{\sum_{cyc}r_a}}\cdot\sqrt{rs^2}=\frac{s}{r.\sqrt{4R+r}}\ge\frac{\sqrt{3r(4R+r)}}{r.\sqrt{4R+r}}$$

$$\left(\because s^2\overset{\text{Gerretsen}}{\ge}12Rr+3r^2+4r(R-2r)\overset{\text{Euler}}{\ge}12Rr+3r^2\right)=\sqrt{\frac{3}{r}}$$

$$\therefore\sum_{cyc}\frac{r_b\cdot\sqrt{r_c}}{r_a^2}\ge\sqrt{\frac{3}{r}}\ \forall\,\Delta\,ABC,''=''\ \text{iff }\Delta\,ABC\text{ is equilateral (QED)}$$