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In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{h_b\cdot\sqrt{h_c}}{h_a^2}\geq\sqrt{\frac{3}{r}}$$

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$$\sum_{cyc}\frac{h_b\cdot\sqrt{h_c}}{h_a^2}=\sum_{cyc}\frac{\left(\frac{1}{h_a}\right)^2}{\frac{1}{h_b}\cdot\sqrt{\frac{1}{h_c}}}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}\frac{1}{h_a}\right)^2}{\sum_{cyc}\left(\frac{1}{h_b}\cdot\sqrt{\frac{1}{h_c}}\right)}\overset{\text{CBS}}{\geq}\frac{\frac{1}{r^2}}{\sqrt{\sum_{cyc}\frac{1}{h_bh_c}}\cdot\sqrt{\sum_{cyc}\frac{1}{h_b}}}$$

$$=\frac{\sqrt{r}}{r^2\cdot\sqrt{\sum_{cyc}h_a}}\cdot\sqrt{\frac{2r^2s^2}{R}}=\frac{\sqrt{r}}{r^2\cdot\sqrt{\frac{\sum_{cyc}ab}{2R}}}\cdot\sqrt{\frac{2r^2s^2}{R}}\geq\frac{\sqrt{3r}}{r^2\cdot\sqrt{\frac{\left(\sum_{cyc}a\right)^2}{2R}}}\cdot\sqrt{\frac{2r^2s^2}{R}}$$

$$=\frac{\sqrt{3r}}{r^2\cdot\sqrt{\frac{4s^2}{2R}}}\cdot\sqrt{\frac{2r^2s^2}{R}}=\sqrt{\frac{3}{r}}\therefore\sum_{cyc}\frac{h_b\cdot\sqrt{h_c}}{h_a^2}\geq\sqrt{\frac{3}{r}}\ \forall\,\Delta\text{ ABC},$$

'' = '' iff Δ ABC is equilateral (QED)