ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$9\sum\tan^2\frac{A}{2}\le\frac{R}{2r}\sum\cot^2\frac{A}{2}$$

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$$\sum\cot^2\frac{A}{2}=\sum\frac{s^2}{r_a^2}=s^2\left(\left(\sum\frac{1}{r_a}\right)^2-2\sum\frac{1}{r_ar_b}\right)=$$

$$=s^2\left(\frac{1}{r^2}-\frac{2(4R+r)}{s^2r}\right)=\frac{s^2-2r^2-8Rr}{r^2}$$

$$\sum\cot^2\frac{A}{2}=\frac{s^2-2r^2-8Rr}{r^2}\overset{Gerretsen}{\ge}\frac{(16Rr-5r^2)-2r^2-8Rr}{r^2}=$$

$$=\frac{8Rr-7r^2}{r^2}=\frac{8R}{r}-7$$

$$\sum\tan^2\frac{A}{2}=\frac{(4R+r)^2}{s^2}-2\overset{Doucet}{\le}\frac{(4R+r)^2}{3r(4R+r)}-2=\frac{4R-5r}{3r}$$

$$We\ need\ to\ show:$$

$$9\sum\tan^2\frac{A}{2}\underset{\frac{R}{r}=x\ge2}{\le}\frac{R}{2r}\sum cot^2\frac{A}{2}\ or,9\times\frac{4R-5r}{3r}\le\left(\frac{8R}{r}-7\right)\frac{R}{2r}$$

$$12x-15\overset{Euler}{\le}(8x-7)\times\frac{x}{2}\ or\ 8x^2-7x\ge24x-30$$

$$8x^2-31x+30\ge0\ or\ (x-2)(8x-15)\ge0\ true\ as\ x\ge2$$

Equality holds for an equilateral triangle.