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In any Δ ABC the following relationship holds :

$$\sum_{cyc}\left(\frac{\cos A}{1+\sin A}\right)^n \ge 3\left(2-\sqrt{3}\right)^n, n\in\mathbb{N}$$

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$$\sum_{cyc}\frac{\cos A}{1+\sin A}=\sum_{cyc}\frac{\cos^2\frac{A}{2}-\sin^2\frac{A}{2}}{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}+2\cos\frac{A}{2}.\sin\frac{A}{2}}$$

$$=\sum_{cyc}\frac{\left(\cos\frac{A}{2}+\sin\frac{A}{2}\right)\left(\cos\frac{A}{2}-\sin\frac{A}{2}\right)}{\left(\cos\frac{A}{2}+\sin\frac{A}{2}\right)^2}=\sum_{cyc}\frac{\cos\frac{A}{2}-\sin\frac{A}{2}}{\cos\frac{A}{2}+\sin\frac{A}{2}}=\sum_{cyc}\frac{1-\tan\frac{A}{2}}{1+\tan\frac{A}{2}}$$

$$=\sum_{cyc}\frac{s-r_a}{s+r_a}=\sum_{cyc}\frac{-s-r_a+2s}{s+r_a}=-3+\sum_{cyc}\frac{2s(s+r_b)(s+r_c)}{(s+r_a)(s+r_b)(s+r_c)}$$

$$=-3+\sum_{cyc}\frac{2s(s^2+s(r_b+r_c)+r_br_c)}{s^3+s^2(4R+r)+s.s^2+rs^2}=-3+\frac{2s(3s^2+2s(4R+r)+s^2)}{2s^3+s^2(4R+r)+rs^2}$$

$$=-3+\frac{4s(2s^2+s(4R+r)+rs-rs)}{s(2s^2+s(4R+r)+rs)}=1-\frac{4r}{2s+4R+2r}=1-\frac{2r}{s+2R+r}$$

$$\overset{?}{\ge}3\left(2-\sqrt{3}\right)\Rightarrow 3\sqrt{3}-5\overset{?}{\ge}\frac{2r}{s+2R+r}\Leftrightarrow\left(3\sqrt{3}-5\right)(s+2R+r)\underbrace{\overset{?}{\ge}}_{(*)}2r$$

$$\text{Now, since } s\overset{\text{Mitrinovic}}{\ge}3\sqrt{3}r \text{ and } R\overset{\text{Euler}}{\ge}2r \text{ and } \because 3\sqrt{3}-5>0$$

$$\therefore \text{LHS of } (*)\ge\left(3\sqrt{3}-5\right)\left(3\sqrt{3}+5\right)r=(27-25)r=2r\Rightarrow(*)\text{ is true}$$

$$\therefore\sum_{cyc}\frac{\cos A}{1+\sin A}\ge3\left(2-\sqrt{3}\right)\text{ and so, }\sum_{cyc}\left(\frac{\cos A}{1+\sin A}\right)^n\overset{\text{Holder}}{\ge}\frac{1}{3^{n-1}}.\left(\sum_{cyc}\frac{\cos A}{1+\sin A}\right)^n$$

$$\ge\frac{1}{3^{n-1}}.\left(3\left(2-\sqrt{3}\right)\right)^n=3\left(2-\sqrt{3}\right)^n\therefore\sum_{cyc}\left(\frac{\cos A}{1+\sin A}\right)^n\ge3\left(2-\sqrt{3}\right)^n$$

$\forall\, n\in\mathbb{N}$ and $\forall\,\Delta$ ABC, "=" iff Δ ABC is equilateral (QED)