ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$9 \le \sum \cot^2 \frac{A}{2} \le \left(\frac{2R}{r} - 1\right)^2$$

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Solution by Tapas Das-India

$$\sum \cot^2 \frac{A}{2} = \sum \frac{s^2}{r_a^2} = s^2\left(\left(\sum \frac{1}{r_a}\right)^2 - 2\sum \frac{1}{r_a r_b}\right) =$$

$$= s^2\left(\frac{1}{r^2} - \frac{2(4R+r)}{s^2 r}\right) = \frac{s^2 - 2r^2 - 8Rr}{r^2}$$

$$\sum \cot^2 \frac{A}{2} = \frac{s^2 - 2r^2 - 8Rr}{r^2} \overset{Gerretsen}{\le} \frac{(4R^2 + 4Rr + 3r^2) - 2r^2 - 8Rr}{r^2} =$$

$$= \frac{(2R-1)^2}{r^2} = \left(\frac{2R}{r} - 1\right)^2$$

$$\sum \cot^2 \frac{A}{2} = \frac{s^2 - 2r^2 - 8Rr}{r^2} \overset{Gerretsen}{\ge} \frac{(16Rr - 5r^2) - 2r^2 - 8Rr}{r^2} =$$

$$= \frac{8Rr - 7r^2}{r^2} = \frac{8R}{r} - 7 \overset{Euler}{\ge} 8 \times 2 - 7 = 9$$

Equality holds for an equilateral triangle.