ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$8 \le \prod \frac{s_b + s_c}{s_a} \le \frac{8}{729}\left(\frac{4R}{r} + 1\right)^3$$

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Lemma:

$$\frac{R}{r} \ge \frac{m_b}{h_c} + \frac{m_c}{h_b}$$

Reference: " New Triangle inequalities with Brocard's Angle " by Bogdan Fustei, Mohamed Amine Ben Ajiba: www.ssmrmh.ro

$$\frac{s_b}{s_c} + \frac{s_c}{s_b} \overset{\substack{s_b \le m_b, s_c \ge h_c \\ s_c \le m_c, s_b \ge h_b}}{\le} \frac{m_b}{h_c} + \frac{m_c}{h_b} \overset{lemma}{\le} \frac{R}{r} \ (1)$$

$$\prod \frac{s_b + s_c}{s_a} = \frac{(s_b + s_c)(s_c + s_a)(s_a + s_b)}{s_a s_b s_c} = \prod \frac{(s_b + s_c)}{\sqrt{s_b s_c}} = \prod\left(\sqrt{\frac{s_b}{s_c}} + \sqrt{\frac{s_c}{s_b}}\right) \le$$

$$\overset{CBS}{\le} \prod \sqrt{2\left(\frac{s_b}{s_c} + \frac{s_c}{s_b}\right)} \overset{(1)}{\le} \prod \sqrt{\frac{2R}{r}} = \left(\sqrt{\left(\frac{2R}{r}\right)}\right)^3 \overset{\frac{R}{r} = x \ge 2}{=} \left(\sqrt{2x}\right)^3$$

We need to show:

$$\left(\sqrt{2x}\right)^3 \le \frac{8}{729}\left(\frac{4R}{r} + 1\right)^3 \overset{\frac{R}{r} = x}{=} \frac{8}{729}(4x + 1)^3 \ \ or, \left(\frac{2}{9}(4x + 1)\right)^3 - \left(\sqrt{2x}\right)^3 \ge 0$$

$$or, \left(\frac{2}{9}(4x + 1) - \sqrt{2x}\right)\left(\left(\frac{2(4x + 1)}{9}\right)^2 + \frac{2(4x + 1)}{9}\sqrt{2x} + \left(\sqrt{2x}\right)^2\right) \overset{\substack{a^3 - b^3 = \\ (a-b)(a^2 + ab + b^2)}}{\ge} 0$$

$$\left(\begin{array}{c} True \ as \ x = \frac{R}{r} \ge 2 \ then \\ \left(\left(\frac{2(4x + 1)}{9}\right)^2 + \frac{2(4x + 1)}{9}\sqrt{2x} + \left(\sqrt{2x}\right)^2\right) > 0 \ and \\ \left(\frac{2}{9}(4x + 1) - \sqrt{2x}\right) \ge \left(\frac{2}{9}(4 \times 2 + 1) - \sqrt{2 \times 2}\right) = 2 - 2 = 0 \end{array}\right)$$

$$\prod \frac{s_b + s_c}{s_a} = \frac{(s_b + s_c)(s_c + s_a)(s_a + s_b)}{s_a s_b s_c} \overset{Cesaro}{\ge} \frac{8 s_a s_b s_c}{s_a s_b s_c} = 8$$

Equality holds for an equilateral triangle.