ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$8 \le \prod \frac{m_b^2 + m_c^2}{m_a^2} \le \frac{8}{729}\left(\frac{2R^2}{r^2} + 1\right)^3$$

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$$\prod \frac{m_b^2 + m_c^2}{m_a^2} = \frac{(m_a^2 + m_b^2)(m_b^2 + m_c^2)(m_c^2 + m_a^2)}{m_a^2 m_b^2 m_c^2} =$$

$$= \frac{(\sum m_a^2)(\sum m_a^2 m_b^2) - m_a^2 m_b^2 m_c^2}{m_a^2 m_b^2 m_c^2} = \sum m_a^2 \frac{(\sum m_a^2 m_b^2)}{m_a^2 m_b^2 m_c^2} - 1 = \sum m_a^2 \sum \frac{1}{m_a^2} - 1 \le$$

$$\overset{m_a \ge \sqrt{s(s-a)}}{\le} \frac{3}{4}\sum a^2 \sum \frac{1}{s(s-a)} - 1 \overset{Leibniz}{\le} \frac{3}{4}.9R^2\left(\frac{4R+r}{s^2 r}\right) - 1 \overset{s^2 \ge 3r(4R+r)}{\le}$$

$$\le \frac{27}{4}R^2.\frac{1}{3r^2} - 1 = \frac{9}{4}\left(\frac{R}{r}\right)^2 - 1$$

$$We\ need\ to\ show:$$

$$\frac{9}{4}\left(\frac{R}{r}\right)^2 - 1 \le \frac{8}{729}\left(\frac{2R^2}{r^2} + 1\right)^3 \quad or \quad \frac{9}{4}x^2 - 1 \overset{\frac{R}{r} = x \ge 2}{\le} \frac{8}{729}(2x^2 + 1)^3$$

$$\frac{9}{4}y - 1 \overset{x^2 = y}{\le} \frac{8}{729}(2y + 1)^3 \ or\ 6561y - 2916 \le 32(2y + 1)^3$$

$$6561y - 2916 \le 32(8y^3 + 12y^2 + 6y + 1)$$

$$256y^3 + 384y^2 - 6369y + 2948 \ge 0$$

$$(y - 4)(256x^4 + 1408x^2 - 737) \ge 0\ this\ is\ true\ as\ y = x^2 = \left(\frac{R}{r}\right)^2 \ge 2^2 = 4$$

$$\prod \frac{m_b^2 + m_c^2}{m_a^2} = \frac{(m_a^2 + m_b^2)(m_b^2 + m_c^2)(m_c^2 + m_a^2)}{m_a^2 m_b^2 m_c^2} \overset{Cesaro}{\ge} \frac{8m_a^2 m_b^2 m_c^2}{m_a^2 m_b^2 m_c^2} = 8$$

Equality holds for an equilateral triangle.