ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{1}{a}\sqrt{\frac{b^2+c^2}{b^2+bc+c^2}}\le\sqrt{2}\,\frac{R}{4r^2}$$

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It is known that:

$$\sum_{cyc}\frac{1}{a^2}\le\frac{1}{4r^2}$$

$$\frac{b^2+c^2}{b^2+bc+c^2}\overset{A-G}{\le}\frac{b^2+c^2}{3bc}=\frac{1}{3}\left(\frac{b}{c}+\frac{c}{b}\right)$$

$$Analogous:$$

$$\frac{b^2+a^2}{a^2+ba+b^2}\overset{A-G}{\le}\frac{1}{3}\left(\frac{a}{b}+\frac{b}{a}\right);\ \frac{c^2+a^2}{a^2+ca+c^2}\overset{A-G}{\le}\frac{1}{3}\left(\frac{a}{c}+\frac{c}{a}\right)\ (1)$$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\overset{CBS}{\le}\sqrt{\sum_{cyc}a^2}\cdot\sqrt{\sum_{cyc}\frac{1}{a^2}}\overset{LEIBNIZ}{\le}\sqrt{9R^2}\cdot\sqrt{\frac{1}{4r^2}}=\frac{3R}{2r}$$

$$Analogous:$$

$$\frac{b}{a}+\frac{a}{c}+\frac{c}{b}\le\frac{3R}{2r}\ (2)$$

$$\sum_{cyc}\frac{1}{a}\sqrt{\frac{b^2+c^2}{b^2+bc+c^2}}\overset{C-B-S}{\le}\sqrt{\sum_{cyc}\frac{1}{a^2}}\cdot\sqrt{\sum_{cyc}\frac{b^2+c^2}{b^2+bc+c^2}}\overset{(1)}{\le}$$

$$\le\sqrt{\frac{1}{4r^2}}\cdot\sqrt{\frac{1}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\frac{1}{3}\left(\frac{b}{a}+\frac{a}{c}+\frac{c}{b}\right)}\overset{(2)}{\le}$$

$$\le\frac{1}{2r}\sqrt{\frac{1}{3}\cdot\frac{3R}{2r}+\frac{1}{3}\cdot\frac{3R}{2r}}=\frac{1}{2r}\sqrt{\frac{R}{r}}=\frac{1}{2r}\sqrt{\frac{4Rr}{4r^2}}=\frac{1}{4r^2}\sqrt{4Rr}\overset{Euler}{\le}\frac{1}{4r^2}\sqrt{2R^2}=\sqrt{2}\,\frac{R}{4r^2}$$

$$Equality\ holds\ for\ a=b=c.$$