ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{w_a}{h_a}+\frac{h_a}{w_a}\le\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}\le min\left\{\frac{w_b}{w_c}+\frac{w_c}{w_b},\frac{2w_a}{h_a}\right\}$$

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$$Let\ u=\cos\left(\frac{B-C}{2}\right)\ and\ v=\cos\left(\frac{B+C}{2}\right)=\sin\frac{A}{2}.$$

$$Also, assume\ M\ to\ be\ the\ middle\ term.$$

$$We\ know\ \frac{|B-C|}{2}<\frac{B+C}{2}<\frac{\pi}{2}\Rightarrow 0<v<u\le 1.\quad Now, let's\ prove\ the\ LHS:$$

$$*\ \frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}=\frac{1+\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}{\frac{1}{2}\left[\cos\left(\frac{B+C}{2}\right)+\cos\left(\frac{B-C}{2}\right)\right]}=\frac{2+2vu}{v+u}.$$

$$The\ inequality\ \frac{w_a}{h_a}+\frac{h_a}{w_a}\le M\ is\ equivalent\ to:$$

$$\frac{w_a}{h_a}+\frac{h_a}{w_a}=\frac{u^2+1}{u}\le M=\frac{2+2vu}{v+u}\Leftrightarrow (u^2-1)(u-v)\le 0$$

$$which\ is\ true\ \because v<u\le 1.$$

$$Now, let\ us\ prove\ the\ RHS\ \ i.e.\ M\le\frac{2w_a}{h_a}=\frac{2}{u}:$$

$$M=\frac{2+2vu}{v+u}\le\frac{2}{u}\Leftrightarrow u(1+vu)\le v+u\Leftrightarrow u^2\le 1\quad which\ is\ true\ \because u\le 1.$$

$$At\ last, WLOG\ assume: b\ge c.$$

$$Let\ t=\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}\le 1. Now,\frac{w_b}{w_c}=\left[\frac{c(a+b)}{b(a+c)}\right]\cdot\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}=k\cdot t\ \ ;\frac{c(a+b)}{b(a+c)}=k$$

$$As\ b\ge c\Rightarrow k=\frac{ac+bc}{ab+bc}\le 1. Thus, 0<kt\le t\le 1.$$

$$We\ can\ observe\ that\ the\ function\ f(x)=x+\frac{1}{x}\ is\ strictly\ decreasing\ on\ (0,1].$$

$\boldsymbol{And\ the\ middle\ term\ is\ f(t), i.e., f(t) = M.}$

$$\mathbf{0} < kt \leq t \leq \mathbf{1} \implies f(kt) \overset{Monotonicity}{\geq} f(t) \implies \frac{w_b}{w_c} + \frac{w_c}{w_b} \geq M.$$

$$\therefore \frac{w_a}{h_a} + \frac{h_a}{w_a} \leq \frac{\cos\frac{B}{2}}{\cos\frac{C}{2}} + \frac{\cos\frac{C}{2}}{\cos\frac{B}{2}} \leq min\left\{\frac{w_b}{w_c} + \frac{w_c}{w_b}, \frac{2w_a}{h_a}\right\}$$

$\boldsymbol{Equality\ holds\ if\ and\ only\ if\ the\ triangle\ is\ isosceles\ (b\ =\ c).}$