ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$s(s-a)+\frac{1}{3}(b-c)^2 \le p_a m_a \le s(s-a)+\frac{1}{2}(b-c)^2$$

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$$s(s-a)+\frac{1}{3}(b-c)^2 \overset{?}{\le} p_a m_a$$

$$\Leftrightarrow s^2(s-a)^2+\frac{(b-c)^4}{9}+\frac{2}{3}s(s-a)(b-c)^2 \overset{?}{\le} \left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)\left(s(s-a)+\frac{(b-c)^2}{4}\right)$$

$$\Leftrightarrow s^2(s-a)^2+\frac{(b-c)^4}{9}+\frac{2}{3}s(s-a)(b-c)^2 \overset{?}{\le} s^2(s-a)^2+s(s-a).\frac{(b-c)^2}{4}+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}.s(s-a)+\frac{s(3s+a)}{(2s+a)^2}.\frac{(b-c)^4}{4}$$

$$\Leftrightarrow \left(\frac{s(3s+a)}{4(2s+a)^2}-\frac{1}{9}\right)(b-c)^2+ s(s-a).\left(\frac{1}{4}+\frac{s(3s+a)}{(2s+a)^2}-\frac{2}{3}\right) \overset{?}{\ge} 0 \ (\because (b-c)^2 \ge 0)$$

$$\Leftrightarrow \frac{(s-a)(11s+4a)}{36(2s+a)^2}(b-c)^2+s(s-a)\left(\frac{(s-a)(16s+8a)+3a^2}{12(2s+a)^2}\right) \overset{?}{\ge} 0 \to \text{true}$$

$$\because s>a \therefore p_a m_a \ge s(s-a)+\frac{1}{3}(b-c)^2$$

$$\text{Again}, p_a m_a \overset{?}{\le} s(s-a)+\frac{1}{2}(b-c)^2$$

$$\Leftrightarrow s^2(s-a)^2+s(s-a).\frac{(b-c)^2}{4}+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}.s(s-a)+\frac{s(3s+a)}{(2s+a)^2}.\frac{(b-c)^4}{4} \overset{?}{\le} s^2(s-a)^2+\frac{(b-c)^4}{4}+s(s-a)(b-c)^2$$

$$\Leftrightarrow \left(\frac{1}{4}-\frac{s(3s+a)}{4(2s+a)^2}\right).(b-c)^2+s(s-a).\left(\frac{3}{4}-\frac{s(3s+a)}{(2s+a)^2}\right) \overset{?}{\ge} 0 \ (\because (b-c)^2 \ge 0)$$

$$\Leftrightarrow \frac{s^2+3sa+a^2}{4(2s+a)^2}.(b-c)^2+s(s-a).\frac{8sa+3a^2}{4(2s+a)^2} \overset{?}{\ge} 0 \to \text{true} \because s>a$$

$$\therefore p_a m_a \le s(s-a)+\frac{1}{2}(b-c)^2 \text{ and so}, s(s-a)+\frac{1}{3}(b-c)^2 \le p_a m_a \le s(s-a)+\frac{1}{2}(b-c)^2 \ \forall \Delta \text{ ABC}, ''='' \text{ iff } b=c \ (\text{QED})$$