ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{|b^2-c^2|}{a} \le \min\left\{4.\sqrt{R(R-2r)},\, 2\sqrt{s^2-4Rr-19r^2},\, 4.\sqrt{\frac{62}{135}(2s^2-27Rr)}\right\}$$

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Let $x = s-a, y = s-b, z = s-c$ & then : $a = y+z, b = z+x, c = x+y$ and $s = x+y+z$ and furthermore, we denote : $\frac{y+z}{x} = m$ and $\frac{yz}{x^2} = n$ and then, we have the following set "S" of relations : $y^2+z^2 = x^2(m^2-2n)$,

$y^3+z^3 = x^3(m^3-3nn), y^4+z^4 = x^4((m^2-2n)^2-2n^2)$,

$y^5+z^5 = x^5\left(m((m^2-2n)^2+n^2-nm^2)\right), y^6+z^6 = x^6((m^3-3nn)^2-2n^3)$,

$$\text{and now, } \frac{|b^2-c^2|}{a} \overset{?}{\le} 4.\sqrt{R(R-2r)}$$

$$\Leftrightarrow \frac{(b+c)^2(b-c)^2}{a^2} \overset{?}{\le} 16\left(\frac{a^2b^2c^2}{16s(s-a)(s-b)(s-c)} - \frac{abc}{2s}\right)$$

$$\Leftrightarrow \frac{(2x+y+z)^2(y-z)^2}{(y+z)^2} \overset{?}{\le} 16.\frac{(y+z)^2(z+x)^2(x+y)^2 - 8xyz(y+z)(z+x)(x+y)}{16xyz(x+y+z)}$$

$$\Leftrightarrow x^4(y^2+z^2)^2 + 12x^4.y^2z^2 + 2x^3(y^5+z^5) - 6x^3.yz(y^3+z^3) + 4x^3.y^2z^2(y+z) + x^2(y^6+z^6) - 5x^2.yz(y^4+z^4) - 9x^2.y^2z^2(y^2+z^2) - 6x^2.y^3z^3 + x.yz(y^5+z^5) + x.y^2z^2(y^3+z^3) - 2x.y^3z^3(y+z) + y^2z^2(y+z)^4 \underset{①}{\overset{?}{\ge}} 0$$

and via set of relations "S", in order to prove ①, it suffices to prove, following simplification :

$$\overbrace{(m^4-4m^3+20m^2+32m+16)}^{\Omega_1}n^2 + \overbrace{(m^5-11m^4-16m^3-4m^2)}^{\Omega_2}n + m^4(m+1)^2 \underset{②}{\overset{?}{\ge}} 0$$

and now, LHS of ② is a quadratic polynomial in "n" with discriminant, $\delta = \Omega_2^2 - 4\Omega_1.m^4(m+1)^2$
$= -m^4(m+1)^2(3m+2)(m+6)(m-2)^2 \le 0 \to$ true ($\because m > 0$)

$$\therefore \boxed{\frac{|b^2-c^2|}{a} \le 4.\sqrt{R(R-2r)}} \text{ and again, } \frac{|b^2-c^2|}{a} \overset{?}{\le} 2\sqrt{s^2-4Rr-19r^2}$$

$$\Leftrightarrow \frac{(2x+y+z)^2(y-z)^2}{(y+z)^2} \overset{?}{\le} \frac{4((x+y+z)^3-(y+z)(z+x)(x+y)-19xyz)}{x+y+z}$$

$$\Leftrightarrow 16x^3.yz + 32x^2.yz(y+z) + 3x(y^4+z^4) - 44x.yz(y^2+z^2) - 94x.y^2z^2 + 3(y+z)^5 \underset{(i)}{\overset{?}{\ge}} 0$$

and via set of relations "S", to prove (i), it suffices to prove,

$$\text{following simplification}: 3m^5+3m^4+n(16+32m-56m^2)\boxed{\overset{?}{\underset{(ii)}{\geq}}}0 \text{ and it's}$$

$$\text{trivially true if}: 16+32m-56m^2\geq 0 \text{ and when}: 16+32m-56m^2<0,$$

$$\text{then, since } n \overset{AM-GM}{\leq} \frac{m^2}{4}, \text{ in order to prove (ii), it suffices to prove}:$$

$$3m^5+3m^4+m^2(4+8m-14m^2)\overset{?}{\geq}0 \Leftrightarrow m^2(3m+1)(m-2)^2\overset{?}{\geq}0$$

$$\rightarrow \text{true} \therefore \boxed{\frac{|b^2-c^2|}{a}\leq 2\sqrt{s^2-4Rr-19r^2}} \;\&\; \frac{|b^2-c^2|}{a}\overset{?}{\leq}4.\sqrt{\frac{62}{135}(2s^2-27Rr)}$$

$$\Leftrightarrow \frac{(2x+y+z)^2(y-z)^2}{(y+z)^2}\overset{?}{\leq}4.\frac{496(x+y+z)^3-1674(y+z)(z+x)(x+y)}{135(x+y+z)}$$

$$\Leftrightarrow 1444x^3(y+z)^2+2160x^3.yz-1824x^2(y+z)^3+4320x^2.yz(y+z)-$$
$$1419x(y^2+z^2)^2-276x.y^2z^2-2976x.yz(y^2+z^2)+1849\left(y^5+z^5\right)+$$
$$3089yz(y^3+z^3)+22y^2z^2(y+z)\boxed{\overset{?}{\underset{(a)}{\geq}}}0 \text{ and via set of relations ''S''},$$

$$\text{in order to prove } (a), \text{ it suffices to prove, following simplification}:$$
$$1849m^5-1419m^4-1824m^3+1444m^2+$$
$$n(2160+4320m+2700m^2-6156m^3)\boxed{\overset{?}{\underset{(b)}{\geq}}}0 \text{ and it's trivially true if}:$$

$$2160+4320m+2700m^2-6156m^3\geq 0 \text{ and when}:$$

$$2160+4320m+2700m^2-6156m^3<0, \text{ then, since } n\overset{AM-GM}{\leq}\frac{m^2}{4},$$

$$\therefore \text{ to prove (b), it suffices to prove}: 1849m^5-1419m^4-1824m^3+1444m^2+$$
$$m^2(540+1080m+675m^2-1539m^3)\overset{?}{\geq}0 \Leftrightarrow 62m^2(5m+8)(m-2)^2\overset{?}{\geq}0$$

$$\rightarrow \text{true } (\because m>0) \therefore \boxed{\frac{|b^2-c^2|}{a}\leq 4.\sqrt{\frac{62}{135}(2s^2-27Rr)}} \text{ and so,}$$

$$\frac{|b^2-c^2|}{a}\leq \min\left\{4.\sqrt{R(R-2r)}, 2\sqrt{s^2-4Rr-19r^2}, 4.\sqrt{\frac{62}{135}(2s^2-27Rr)}\right\}$$

$$\forall\,\Delta\,ABC, ''='' \text{ for } 1^{st} \text{ inequality iff } b+c=2a \text{ and } ''='' \text{ for}$$
$$2^{nd} \text{ and } 3^{rd} \text{ inequalities iff } \Delta\,ABC \text{ is equilateral (QED)}$$