ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\mathbf{s(s-a)+\frac{2}{3}(b-c)^2 \le p_a n_a}$$

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$$p_a n_a \overset{?}{\ge} s(s-a)+\frac{2}{3}(b-c)^2$$

$$\Leftrightarrow \left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)\left(s(s-a)+\frac{s}{a}(b-c)^2\right) \overset{?}{\ge}$$

$$s^2(s-a)^2+\frac{4(b-c)^4}{9}+\frac{4}{3}s(s-a)(b-c)^2$$

$$\Leftrightarrow s^2(s-a)^2+\frac{s^2(s-a)}{a}(b-c)^2+\frac{s^2(3s+a)(s-a)}{(2s+a)^2}(b-c)^2+$$

$$\frac{s^2(3s+a)}{a(2s+a)^2}(b-c)^4 \overset{?}{\ge} s^2(s-a)^2+\frac{4(b-c)^4}{9}+\frac{4}{3}s(s-a)(b-c)^2$$

$$\Leftrightarrow \left(\frac{s^2(3s+a)}{a(2s+a)^2}-\frac{4}{9}\right)(b-c)^2+s(s-a)\left(\frac{s}{a}+\frac{s(3s+a)}{(2s+a)^2}-\frac{4}{3}\right) \overset{?}{\ge} 0$$

$$(\because (b-c)^2 \ge 0) \Leftrightarrow \frac{(s-a)(27s^2+20sa+4a^2)}{9a(2s+a)^2}.(b-c)^2+$$

$$s(s-a).\frac{(s-a)(12s^2+17sa+7a^2)+3a^3}{3a(2s+a)^2} \overset{?}{\ge} 0 \to \text{true} \because s > a$$

$$\therefore s(s-a)+\frac{2}{3}(b-c)^2 \le p_a n_a \ \forall \Delta \text{ ABC}, ''='' \text{ iff } b=c \text{ (QED)}$$