ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship ho$l$ds :

$$g_a n_a \le s(s-a)+\frac{1}{2}(b-c)^2 \le m_a n_a$$

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$$g_a n_a \overset{?}{\le} s(s-a)+\frac{1}{2}(b-c)^2$$

$$\Leftrightarrow \left(s(s-a)-\frac{(s-a)(b-c)^2}{a}\right)\left(s(s-a)+\frac{s}{a}(b-c)^2\right) \overset{?}{\le}$$

$$s^2(s-a)^2+\frac{(b-c)^4}{4}+s(s-a)(b-c)^2$$

$$\Leftrightarrow s^2(s-a)^2+\frac{s^2(s-a)}{a}(b-c)^2-\frac{s(s-a)^2}{a}(b-c)^2-\frac{s(s-a)}{a^2}(b-c)^4 \overset{?}{\le}$$

$$s^2(s-a)^2+\frac{(b-c)^4}{4}+s(s-a)(b-c)^2$$

$$\Leftrightarrow \left(\frac{1}{4}+\frac{s(s-a)}{a^2}\right)(b-c)^2+s(s-a)\left(1+\frac{s-a}{a}-\frac{s}{a}\right) \overset{?}{\ge} 0 \;(\because (b-c)^2 \ge 0)$$

$$\Leftrightarrow \frac{(2s-a)^2(b-c)^2}{4a^2} \overset{?}{\ge} 0 \to \text{true} \therefore g_a n_a \le s(s-a)+\frac{1}{2}(b-c)^2$$

$$\text{and } m_a n_a \overset{?}{\ge} s(s-a)+\frac{1}{2}(b-c)^2$$

$$\Leftrightarrow \left(s(s-a)+\frac{(b-c)^2}{4}\right)\left(s(s-a)+\frac{s}{a}(b-c)^2\right) \overset{?}{\ge}$$

$$s^2(s-a)^2+\frac{(b-c)^4}{4}+s(s-a)(b-c)^2$$

$$\Leftrightarrow s^2(s-a)^2+\frac{s^2(s-a)}{a}(b-c)^2+\frac{s(s-a)}{4}(b-c)^2+\frac{s}{4a}(b-c)^4 \overset{?}{\ge}$$

$$s^2(s-a)^2+\frac{(b-c)^4}{4}+s(s-a)(b-c)^2$$

$$\Leftrightarrow \frac{s-a}{4a}.(b-c)^2+s(s-a)\left(\frac{s}{a}+\frac{1}{4}-1\right) \overset{?}{\ge} 0 \;(\because (b-c)^2 \ge 0)$$

$$\Leftrightarrow \frac{s-a}{4a}.(b-c)^2+s(s-a)\left(\frac{4s-3a}{a}\right) \overset{?}{\ge} 0 \to \text{true} \because s > a$$

$$\therefore m_a n_a \ge s(s-a)+\frac{1}{2}(b-c)^2 \text{ and so,}$$

$$g_a n_a \le s(s-a)+\frac{1}{2}(b-c)^2 \le m_a n_a \forall \Delta ABC, ''='' \text{ iff } b=c \text{ (QED)}$$