ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$18p_a \geq 7n_a + 4m_a + 7g_a$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$7n_a + 4m_a + 7g_a \overset{\text{CBS}}{\leq} \sqrt{(7+4+7)(7n_a^2 + 4m_a^2 + 7g_a^2)} \overset{?}{\leq} 18p_a$$

$$\Leftrightarrow 18p_a^2 \overset{?}{\geq} 7n_a^2 + 4m_a^2 + 7g_a^2$$

$$\Leftrightarrow 18s(s-a) + \frac{18s(3s+a)(b-c)^2}{(2s+a)^2} \overset{?}{\geq} 7s(s-a) + \frac{7s}{a}(b-c)^2 + 4s(s-a) +$$

$$(b-c)^2 + 7s(s-a) - \frac{7(s-a)(b-c)^2}{a}$$

(via Bogdan Fustei and Mohamed Amine Ben Ajiba and via Bogdan Fustei)

$$\Leftrightarrow \frac{18s(3s+a)}{(2s+a)^2} \overset{?}{\geq} \frac{7s}{a} + 1 - \frac{7(s-a)}{a} \; (\because (b-c)^2 \geq 0) \Leftrightarrow \frac{9s(3s+a)}{(2s+a)^2} \overset{?}{\geq} 4$$

$$\Leftrightarrow 11s^2 - 7sa - 4a^2 \overset{?}{\geq} 0 \Leftrightarrow (s-a)(11s+4a) \overset{?}{\geq} 0 \rightarrow \text{true} \because s > a$$

$$\therefore 18p_a \geq 7n_a + 4m_a + 7g_a \; \forall \Delta \, ABC, '' ='' \text{ iff } b = c \text{ (QED)}$$