ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$16m_a \geq 9p_a + 2g_a + 5\sqrt{r_b r_c}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$9p_a + 2g_a + 5\sqrt{r_b r_c} \overset{CBS}{\leq} \sqrt{(9+2+5)\left(9p_a^2 + 2g_a^2 + 5s(s-a)\right)} \overset{?}{\leq} 16m_a$$

$$\Leftrightarrow 16m_a^2 \overset{?}{\geq} 9p_a^2 + 2g_a^2 + 5s(s-a)$$

$$\Leftrightarrow 16s(s-a) + 4(b-c)^2 \overset{?}{\geq} 9s(s-a) + \frac{9s(3s+a)(b-c)^2}{(2s+a)^2} +$$

$$2s(s-a) - \frac{2(s-a)(b-c)^2}{a} + 5s(s-a)$$

(via Bogdan Fustei and Mohamed Amine Ben Ajiba and via Bogdan Fustei)

$$\Leftrightarrow 4 + \frac{2(s-a)}{a} \overset{?}{\geq} \frac{9s(3s+a)}{(2s+a)^2} \; (\because (b-c)^2 \geq 0) \Leftrightarrow 8s^3 - 11s^2a + sa^2 + 2a^3 \overset{?}{\geq} 0$$

$$\Leftrightarrow (s-a)\left((s-a)(8s+5a) + 3a^2\right) \overset{?}{\geq} 0 \to \text{true} \because s > a$$

$$\therefore 16m_a \geq 9p_a + 2g_a + 5\sqrt{r_b r_c} \; \forall \, \Delta \, ABC, '' ='' \text{ iff } b = c \text{ (QED)}$$