ROMANIAN MATHEMATICAL MAGAZINE

In $any\ \Delta$ ABC holds :

$$max\left\{\frac{w_b+w_c}{h_b+h_c}+\frac{h_b+h_c}{w_b+w_c},\frac{m_b+m_c}{w_b+w_c}+\frac{w_b+w_c}{m_b+m_c}\right\}\le\frac{m_b+m_c}{h_b+h_c}+\frac{h_b+h_c}{m_b+m_c}\le\frac{\sqrt{s^2-7Rr+51r^2}}{4r}$$

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Let $x=s-a, y=s-b, z=s-c$; then : $a=y+z, b=z+x, c=x+y$ and $s=x+y+z$ and furthermore, we denote : $\frac{y+z}{x}=m$ and $\frac{yz}{x^2}=n$ and then, we have the following set "S" of relations : $y^2+z^2=x^2(m^2-2n)$, $y^3+z^3=x^3(m^3-3nn), y^4+z^4=x^4((m^2-2n)^2-2n^2)$, $y^5+z^5=x^5\left(m((m^2-2n)^2+n^2-nm^2)\right), y^6+z^6=x^6((m^3-3nn)^2-2n^3)$,

$$\text{and now, } (m_b+m_c)^2\le m_b^2+m_c^2+\frac{2a^2+bc}{2}$$

(Reference : Solution to Inequality in Triangle – 316 by Dang Ngoc Minh;)

$$=\frac{4s(s-b)+(c-a)^2+4s(s-c)+(a-b)^2+2(2a^2+bc)}{4}\overset{?}{\le}$$

$$\frac{s^2-7Rr+3r^2}{16r^2}.(h_b+h_c)^2=\frac{s^3-7Rrs+3r^2s}{16r^2}\left(4r^2s\left(\frac{1}{b^2}+\frac{1}{c^2}+\frac{2}{bc}\right)\right)$$

$$\Leftrightarrow 4(y+z)\left(\sum_{cyc}x\right)+(z-x)^2+(x-y)^2+2\left(2(y+z)^2+(z+x)(x+y)\right)\overset{?}{\le}$$

$$\frac{(x+y+z)(2x+y+z)^2}{(z+x)^2(x+y)^2}\left((x+y+z)^3-\frac{7}{4}(y+z)(z+x)(x+y)+3xyz\right)$$

$$\Leftrightarrow 4x^5(y+z)-4x^4(y+z)^2+16x^4.yz-3x^3(y+z)^3+4x^3.yz(y+z)+$$
$$26x^2(y^2+z^2)^2-112x^2.y^2z^2+4x^2.yz(y^2+z^2)+25x(y^5+z^5)+$$
$$30xyz(y^3+z^3)-51x.y^2z^2(y+z)+4(y^6+z^6)+17yz(y^4+z^4)-4y^2z^2(y+z)^3$$
$$-26y^3z^3\ \boxed{\overset{?}{\underset{(1)}{\ge}}}\ 0$$

and via set of relations "S", to prove (1), suffices to prove,

$$\overbrace{(36m^2+16m+16)}^{\sigma_1}n^2+\overbrace{(7m^4+95m^3+100m^2-4m-16)}^{\sigma_2}n-$$
$$\overbrace{(4m^6+25m^5+26m^4-3m^3-4m^2+4m)}^{\sigma_3}\ \boxed{\overset{?}{\underset{(2)}{\ge}}}\ 0$$

; now, discriminant, $\boxed{\delta}=$

$$\sigma_2^2+4\sigma_1\sigma_3=(m+1)^2(m+2)^2\overbrace{(625m^4+1436m^3-716m^2-96m+64)}^{\mu}\ \&$$

$$\mu=\frac{1}{27}\left((1875m^2+5558m+1349)(3m-1)^2+379-56m\right)>0\text{ for }m\le\frac{379}{56}$$

$$\&\ \mu = (625m^2 + 2686m + 4031)(m-1)^2 + 5280m - 3967 > 0 \ \textit{when} : m > \frac{379}{56}$$

$\boxed{\delta} > 0\ \forall\ m > 0$ *and so, in order to prove* ②, it suffices to prove :

$$2\sigma_1.n \boxed{\underset{(m)}{\overset{?}{\leq}}} -\sigma_2 + \sqrt{\delta} \text{ AND } 2\sigma_1.n \boxed{\underset{(n)}{\overset{?}{\geq}}} -\sigma_2 - \sqrt{\delta}$$

$$\text{Since } n \overset{AM-GM}{\leq} \frac{m^2}{4} \therefore \text{ to prove } (m), \text{it suffices to prove} : \sigma_1.\frac{m^2}{2} \overset{?}{\leq} -\sigma_2 + \sqrt{\delta}$$

$$\Leftrightarrow (m+2)^2(25m^2+3m-4) \overset{?}{\leq} \sqrt{\delta} \text{ and it's trivially true if} :$$

$25m^2 + 3m - 4 \leq 0$ *and when* : $25m^2 + 3m - 4 > 0$, *then it suffices to prove* :

$$(m+2)^4(25m^2+3m-4)^2 \overset{?}{\leq} (m+1)^2(m+2)^2\mu \Leftrightarrow$$

$$\boxed{4m(m+2)^2(m-2)^2(9m^2+4m+4) \overset{?}{\geq} 0} \to \text{true}; \ \textit{also}, 2\sigma_1.n + \sqrt{\delta} > \sqrt{\delta}$$

$$= \sqrt{\sigma_2^2 + 4(36m^2+16m+16).m(m+1)^2\left(4m^3 + \overbrace{17m^2 - 12m + 4}^{\Delta = -128 < 0}\right)} > \sqrt{\sigma_2^2}$$

$$\geq -\sigma_2 \Rightarrow 2\sigma_1.n > -\sigma_2 - \sqrt{\delta} \therefore (m),(n) \text{ true} \therefore \frac{m_b + m_c}{h_b + h_c} \overset{(\blacksquare)}{\leq} \frac{\sqrt{s^2 - 7Rr + 3r^2}}{4r}$$

$$\text{Let } \theta = \frac{\sqrt{s^2 - 7Rr + 51r^2}}{4r} \textit{ and then} : \theta^2 \overset{?}{\geq} 4 \Leftrightarrow s^2 - 7Rr + 51r^2 \overset{?}{\geq} 64r^2$$

$$\to \text{true via Gerretsen and Euler} \therefore \theta \geq 2 \to (\blacksquare\blacksquare) \textit{ and let } t = \frac{m_b + m_c}{h_b + h_c}$$

$$\textit{and then} : t + \frac{1}{t} \overset{?}{\leq} \theta \Leftrightarrow 2t \underset{(i)}{\overset{?}{\leq}} \theta + \sqrt{\theta^2 - 4} \text{ AND } 2t \underset{(ii)}{\overset{?}{\geq}} \theta - \sqrt{\theta^2 - 4}$$

$$\text{Now}, 2t \overset{\text{via }(\blacksquare)}{\leq} 2.\frac{\sqrt{s^2 - 7Rr + 3r^2}}{4r} \overset{?}{\leq} \frac{\sqrt{s^2 - 7Rr + 51r^2}}{4r} + \frac{\sqrt{s^2 - 7Rr - 13r^2}}{4r}$$

$$\overset{\text{squaring}}{\Leftrightarrow} s^2 - 7Rr - 13r^2 \overset{?}{\leq} \sqrt{(s^2 - 7Rr - 13r^2)(s^2 - 7Rr + 51r^2)}$$

$$\Leftrightarrow s^2 - 7Rr - 13r^2 \overset{?}{\leq} s^2 - 7Rr + 51r^2 \left(\because s^2 - 7Rr - 13r^2 \overset{\text{Gerretsen and Euler}}{\geq} 0\right)$$

$$\to \text{true} \Rightarrow (i) \text{ is true}; \textit{again}, 2t \geq 2 \overset{?}{\geq} \theta - \sqrt{\theta^2 - 4} \Leftrightarrow \theta^2 - 4 \overset{?}{\geq} \theta^2 - 4\theta + 4 \to$$

$$\text{true via } (\blacksquare\blacksquare) \Rightarrow (ii) \text{ true} \therefore t + \frac{1}{t} \leq \theta \ \&\ \textit{let } \frac{w_b + w_c}{h_b + h_c} = \alpha \geq 1 \ \&\ \frac{m_b + m_c}{w_b + w_c} = \beta \geq 1$$

$$\text{then} : \left(\alpha + \frac{1}{\alpha}\right), \left(\beta + \frac{1}{\beta}\right) \leq t + \frac{1}{t}\left(\because \alpha, \beta \leq t \text{ with } \alpha, \beta, t \geq 1 \ \&k + \frac{1}{k} \text{ is } \uparrow \text{ on } [1, \infty)\right)$$

$$\therefore \left(\alpha + \frac{1}{\alpha}\right), \left(\beta + \frac{1}{\beta}\right) \leq t + \frac{1}{t} \ \&\ \text{so}, \max\left\{\alpha + \frac{1}{\alpha}, \beta + \frac{1}{\beta}\right\} \leq t + \frac{1}{t} \leq \frac{\sqrt{s^2 - 7Rr + 51r^2}}{4r}$$

$\forall\ \Delta$ ABC, " = " iff $a = b = c$ (QED)