ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$m_b + m_c \le \frac{\sqrt{s^2 - 7Rr + 3r^2}}{4r}(h_b + h_c)$$

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Let $x = s - a, y = s - b, z = s - c$; then : $a = y + z, b = z + x, c = x + y$
and $s = x + y + z$ and furthermore, we denote : $\frac{y+z}{x} = m$ and $\frac{yz}{x^2} = n$
and then, we have the following set "S" of relations : $y^2 + z^2 = x^2(m^2 - 2n)$,
$y^3 + z^3 = x^3(m^3 - 3nn), y^4 + z^4 = x^4((m^2 - 2n)^2 - 2n^2)$,
$y^5 + z^5 = x^5\left(m((m^2 - 2n)^2 + n^2 - nm^2)\right), y^6 + z^6 = x^6((m^3 - 3nn)^2 - 2n^3)$,

and now, $(m_b + m_c)^2 \le m_b^2 + m_c^2 + \frac{2a^2 + bc}{2}$

(Reference : Solution to Inequality in Triangle – 316 by Dang Ngoc Minh; published at www.ssmrmh.ro)

$$= \frac{4s(s-b) + (c-a)^2 + 4s(s-c) + (a-b)^2 + 2(2a^2 + bc)}{4} \overset{?}{\le}$$

$$\frac{s^2 - 7Rr + 3r^2}{16r^2}.(h_b + h_c)^2 = \frac{s^3 - 7Rrs + 3r^2s}{16r^2}\left(4r^2s\left(\frac{1}{b^2} + \frac{1}{c^2} + \frac{2}{bc}\right)\right)$$

$$\Leftrightarrow 4(y+z)\left(\sum_{cyc} x\right) + (z-x)^2 + (x-y)^2 + 2\left(2(y+z)^2 + (z+x)(x+y)\right) \overset{?}{\le}$$

$$\frac{(x+y+z)(2x+y+z)^2}{(z+x)^2(x+y)^2}\left(\left(\sum_{cyc} x\right)^3 - \frac{7}{4}(y+z)(z+x)(x+y) + 3xyz\right)$$

$$\Leftrightarrow 4x^5(y+z) - 4x^4(y+z)^2 + 16x^4.yz - 3x^3(y+z)^3 + 4x^3.yz(y+z) +$$
$$26x^2(y^2+z^2)^2 - 112x^2.y^2z^2 + 4x^2.yz(y^2+z^2) + 25x(y^5+z^5) +$$
$$30xyz(y^3+z^3) - 51x.y^2z^2(y+z) + 4(y^6+z^6) + 17yz(y^4+z^4) - 4y^2z^2(y+z)^3$$
$$-26y^3z^3 \overset{?}{\underset{①}{\ge}} 0$$ and via set of relations "S", in order to prove ①,

it suffices to prove, following simplification :

$$\overbrace{(36m^2 + 16m + 16)}^{\sigma_1}n^2 + \overbrace{(7m^4 + 95m^3 + 100m^2 - 4m - 16)}^{\sigma_2}n -$$
$$\overbrace{(4m^6 + 25m^5 + 26m^4 - 3m^3 - 4m^2 + 4m)}^{\sigma_3} \overset{?}{\underset{②}{\le}} 0$$

Now, LHS of ② is a quadratic polynomial in "n" with discriminant, $\boxed{\delta} =$
$\sigma_2^2 + 4\sigma_1\sigma_3 = (m+1)^2(m+2)^2(625m^4 + 1436m^3 - 716m^2 - 96m + 64)$
Now, $625m^4 + 1436m^3 - 716m^2 - 96m + 64 =$

$$\frac{1}{27}\Big((1875m^2+5558m+1349)(3m-1)^2+379-56m\Big)>0 \text{ whenever}:$$
$$0<m\le\frac{379}{56} \text{ and } 625m^4+1436m^3-716m^2-96m+64=$$
$$(625m^2+2686m+4031)(m-1)^2+5280m-3967>0 \text{ when}: m>\frac{379}{56}$$
$$\text{and so}, \boxed{\delta}=(m+1)^2(m+2)^2(625m^4+1436m^3-716m^2-96m+64)$$
$$>0\ \forall\ m>0 \text{ and so, in order to prove ②, it suffices to prove}:$$
$$2\sigma_1.n \underset{(m)}{\overset{?}{\le}} -\sigma_2+\sqrt{\delta} \text{ AND } 2\sigma_1.n \underset{(n)}{\overset{?}{\ge}} -\sigma_2-\sqrt{\delta}$$
$$\text{Now, since } n \overset{AM-GM}{\le} \frac{m^2}{4} \therefore \text{in order to prove (m), it suffices to prove}:$$
$$\sigma_1.\frac{m^2}{2}\overset{?}{\le}-\sigma_2+\sqrt{\delta}\Leftrightarrow(m+2)^2(25m^2+3m-4)\overset{?}{\le}\sqrt{\delta} \text{ and it's trivially true if}:$$
$$25m^2+3m-4\le 0 \text{ and when}: 25m^2+3m-4>0, \text{then it suffices to prove}:$$
$$(m+2)^4(25m^2+3m-4)^2\overset{?}{\le}(m+1)^2(m+2)^2\begin{pmatrix}625m^4+1436m^3-716m^2-\\96m+64\end{pmatrix}$$
$$\Leftrightarrow \boxed{4m(m+2)^2(m-2)^2(9m^2+4m+4)\overset{?}{\ge}0}\to \text{true}$$
$$\because m>0\Rightarrow \text{(m) is true and also}, 2\sigma_1.n+\sqrt{\delta}>\sqrt{\delta}=\sqrt{\sigma_2^2+4\sigma_1\sigma_3}$$
$$=\sqrt{\sigma_2^2+4(36m^2+16m+16).m(m+1)^2(4m^3+17m^2-12m+4)}$$
$$>\sqrt{\sigma_2^2}\ (\because \Delta \text{ of } 17m^2-12m+4=-128<0\Rightarrow 4m^3+17m^2-12m+4>0)$$
$$=|\sigma_2|\ge-\sigma_2\ (\because |x|+x\ge 0\ \forall\ x\in\mathbb{R})\Rightarrow 2\sigma_1.n>-\sigma_2-\sqrt{\delta}\Rightarrow \text{(n) is true}$$
$$\text{(strict inequality)} \therefore \text{(m) and (n) are both true} \Rightarrow ② \Rightarrow ① \text{ is true}$$
$$\therefore m_b+m_c\le\frac{\sqrt{s^2-7Rr+3r^2}}{4r}(h_b+h_c)\ \forall\ \Delta ABC,$$
$$''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$