ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$m_b h_c + m_c h_b \le 2m_a r_a$$

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$$m_a m_b \overset{?}{\le} \frac{2c^2+ab}{4} \Leftrightarrow \left(\frac{2b^2+2c^2-a^2}{4}\right)\left(\frac{2c^2+2a^2-b^2}{4}\right) \overset{?}{\le} \frac{(2c^2+ab)^2}{16}$$

$$\Leftrightarrow a^4+b^4-2a^2b^2-a^2c^2+2abc^2-b^2c^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow (a+b)^2(a-b)^2-c^2(a-b)^2 \overset{?}{\ge} 0 \Leftrightarrow (a-b)^2(a+b+c)(a+b-c) \overset{?}{\ge} 0$$

$$\to \text{true} \Rightarrow m_a m_b \le \frac{2c^2+ab}{4} \text{ and analogs}$$

$$\therefore (m_b h_c + m_c h_b)^2 = m_b^2 h_c^2 + m_c^2 h_b^2 + 2m_b m_c h_b h_c \le$$

$$\frac{(4s(s-b)+(c-a)^2)}{4}.\frac{4r^2s^2}{c^2} + \frac{b^2(4s(s-c)+(a-b)^2)}{4}.\frac{4r^2s^2}{b^2} +$$

$$2\left(\frac{2a^2+bc}{4}\right).\frac{4r^2s^2}{bc} \overset{?}{\le} 4m_a^2r_a^2 = (4s(s-a)+(b-c)^2).\frac{r^2s^2}{(s-a)^2}$$

$$\Leftrightarrow (z+x)^2(x+y)^2\left(4x\left(\sum_{cyc} x\right)+(y-z)^2\right) \overset{?}{\ge}$$

$$x^2(z+x)^2\left(4y\left(\sum_{cyc} x\right)+(z-x)^2\right)+x^2(x+y)^2\left(4z\left(\sum_{cyc} x\right)+(x-y)^2\right)+$$

$$2(z+x)(x+y)x^2\left(2(y+z)^2+(z+x)(x+y)\right)$$

$$\left(\begin{matrix} x=s-a, y=s-b, z=s-c \Rightarrow a=y+z, b=z+x, c=x+y \\ \text{and } s=x+y+z \end{matrix}\right)$$

$$\Leftrightarrow 4x^5(y+z)+5x^4(y+z)^2-20x^4.yz+2x^3\left((y+z)^3-3yz(y+z)\right)-$$

$$10x^3.yz(y+z)+2x^2.yz((y+z)^2-4yz)+2xyz(y+z)^3-$$

$$4xy^2z^2(y+z)+y^2z^2(y-z)^2 \overset{?}{\ge} 0 \text{ and } \because y^2z^2(y-z)^2 \ge 0 \therefore \text{it suffices to prove :}$$

$$4x^5(mx)+5x^4(mx)^2-20x^4.nx^2+2x^3\left((mx)^3-3nx^2(mx)\right)-10x^3.nx^2(mx)$$

$$+2x^2.nx^2((mx)^2-4nx^2)+2xnx^2(mx)^3-4x.n^2x^4(mx) \overset{?}{\ge} 0 \left(\begin{matrix} \frac{y+z}{x}=m \text{ and} \\ \frac{yz}{x^2}=n \end{matrix}\right)$$

$$\Leftrightarrow (8+4m)n^2-(2m^3+2m^2-16m-20)n-(2m^3+5m^2+4m) \boxed{\overset{?}{\underset{(1)}{\le}}} 0$$

Now, LHS of (1) is a quadratic polynomial in "n" with discriminant, $\delta = (2m^3+2m^2-16m-20)^2+4(8+4m)(2m^3+5m^2+4m) > 0$ and so, in order to prove (1), it suffices to prove :

$$2(8+4m)n \underset{(*)}{\overset{?}{\leq}} 2m^3+2m^2-16m-20+\sqrt{\delta} \text{ AND}$$

$$2(8+4m)n \underset{(**)}{\overset{?}{\geq}} 2m^3+2m^2-16m-20-\sqrt{\delta}$$

$$\text{Since } n \overset{AM-GM}{\leq} \frac{m^2}{4} \therefore \text{ in order to prove } (*), \text{it suffices to prove :}$$

$$2(2+m)m^2 \overset{?}{\leq} 2m^3+2m^2-16m-20+\sqrt{\delta}$$

$$\Leftrightarrow 2m^2+16m+20 \overset{?}{\leq} \sqrt{\delta} \overset{\text{squaring}}{\Leftrightarrow}$$

$$(2m^3+2m^2-16m-20)^2+4(8+4m)(2m^3+5m^2+4m) \overset{?}{\geq}$$

$$(2m^2+16m+20)^2 \Leftrightarrow \boxed{4m(m+2)^3(m-2)^2 \overset{?}{\geq} 0} \to \text{true} \because m>0$$

$$\Rightarrow (*) \text{ is true and again, } 2(8+4m)n+\sqrt{\delta} >$$

$$\sqrt{(2m^3+2m^2-16m-20)^2+4(8+4m)(2m^3+5m^2+4m)} >$$

$$\sqrt{(2m^3+2m^2-16m-20)^2} = |2m^3+2m^2-16m-20| \geq$$

$$2m^3+2m^2-16m-20 \text{ and so, } 2(8+4m)n > 2m^3+2m^2-16m-20-\sqrt{\delta}$$

$$\Rightarrow (**) \text{ is true (strict inequality)} \therefore \text{both } (*) \text{ and } (**) \text{ are true} \Rightarrow ① \text{ is true}$$

$$\therefore m_b h_c + m_c h_b \leq 2m_a r_a \ \forall \ \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$