

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$\frac{m_b}{h_c} + \frac{m_c}{h_b} \leq \frac{\sqrt{s^2 - 11r^2}}{2r}$$

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$$\begin{aligned} m_a m_b &\stackrel{?}{\leq} \frac{2c^2 + ab}{4} \Leftrightarrow \left(\frac{2b^2 + 2c^2 - a^2}{4}\right) \left(\frac{2c^2 + 2a^2 - b^2}{4}\right) \stackrel{?}{\leq} \frac{(2c^2 + ab)^2}{16} \\ &\Leftrightarrow a^4 + b^4 - 2a^2b^2 - a^2c^2 + 2abc^2 - b^2c^2 \stackrel{?}{\geq} 0 \\ &\Leftrightarrow (a+b)^2(a-b)^2 - c^2(a-b)^2 \stackrel{?}{\geq} 0 \Leftrightarrow (a-b)^2(a+b+c)(a+b-c) \stackrel{?}{\geq} 0 \\ \rightarrow \text{true} &\Rightarrow m_a m_b \leq \frac{2c^2 + ab}{4} \text{ and analogs } \therefore \left(\frac{m_b}{h_c} + \frac{m_c}{h_b}\right)^2 = \frac{m_b^2}{h_c^2} + \frac{m_c^2}{h_b^2} + \frac{2m_b m_c}{h_b h_c} \leq \\ &\frac{c^2(4s(s-b) + (c-a)^2)}{16r^2 s^2} + \frac{b^2(4s(s-c) + (a-b)^2)}{16r^2 s^2} + \frac{2bc \left(\frac{2a^2+bc}{4}\right)}{4r^2 s^2} \stackrel{?}{\leq} \frac{s^2 - 11r^2}{4r^2} \\ &\Leftrightarrow 4 \left(\sum_{\text{cyc}} x\right)^4 - 44xyz \left(\sum_{\text{cyc}} x\right) \stackrel{?}{\geq} 4 \left(\sum_{\text{cyc}} x\right) y(x+y)^2 + (x+y)^2(z-x)^2 + \\ &4 \left(\sum_{\text{cyc}} x\right) z(z+x)^2 + (z+x)^2(x-y)^2 + 4(y+z)^2(z+x)(x+y) + \\ &2(z+x)^2(x+y)^2 \left(\begin{array}{l} x = s - a, y = s - b, z = s - c \Rightarrow \\ a = y + z, b = z + x, c = x + y \text{ and } s = x + y + z \end{array} \right) \\ &\Leftrightarrow 2x^3(y+z) + x^2(y+z)^2 - 5x^2 \cdot yz - 5x \cdot yz(y+z) + 2yz(y+z)^2 - y^2z^2 \stackrel{?}{\geq} 0 \\ &\Leftrightarrow 2x^3(mx) + x^2(mx)^2 - 5x^2 \cdot nx^2 - 5x \cdot nx^2 \cdot mx + 2nx^2(mx)^2 - n^2x^4 \stackrel{?}{\geq} 0 \\ &\left(\frac{y+z}{x} = m \text{ and } \frac{yz}{x^2} = n\right) \Leftrightarrow n^2 - (2m^2 - 5m - 5)n - (m^2 + 2m) \stackrel{?}{\geq} 0 \end{aligned}$$

Now, LHS of ① is a quadratic polynomial in "n" with discriminant, $\delta = (2m^2 - 5m - 5)^2 + 4(m^2 + 2m) > 0$ and so,

in order to prove ①, it suffices to prove :

$$2n \stackrel{?}{\geq} 2m^2 - 5m - 5 + \sqrt{\delta} \text{ AND } 2n \stackrel{?}{\geq} 2m^2 - 5m - 5 - \sqrt{\delta}$$

Since $n \stackrel{\text{AM-GM}}{\leq} \frac{m^2}{4} \therefore$ in order to prove (*), it suffices to prove :

$$\frac{m^2}{2} \stackrel{?}{\leq} 2m^2 - 5m - 5 + \sqrt{\delta} \Leftrightarrow 2\sqrt{\delta} \stackrel{?}{\geq} 10m + 10 - 3m^2 \text{ and it's trivially true if :}$$

ROMANIAN MATHEMATICAL MAGAZINE

$10m + 10 - 3m^2 \leq 0$ and when : $10m + 10 - 3m^2 > 0$, then it suffices to prove :

$$4(2m^2 - 5m - 5)^2 + 16(m^2 + 2m) \stackrel{?}{\geq} (10m + 10 - 3m^2)^2$$

$$\Leftrightarrow \boxed{m(7m + 8)(m - 2)^2 \stackrel{?}{\geq} 0} \rightarrow \text{true} \because m > 0 \Rightarrow (*) \text{ is true}$$

Again, $2n + \sqrt{\delta} > \sqrt{(2m^2 - 5m - 5)^2 + 4(m^2 + 2m)} > \sqrt{(2m^2 - 5m - 5)^2}$

$$= |2m^2 - 5m - 5| \geq 2m^2 - 5m - 5 \text{ and so, } 2n > 2m^2 - 5m - 5 - \sqrt{\delta}$$

$\Rightarrow (**)$ is true (strict inequality) \therefore both $(*)$ and $(**)$ are true \Rightarrow ① is true

$$\therefore \frac{m_b}{h_c} + \frac{m_c}{h_b} \leq \frac{\sqrt{s^2 - 11r^2}}{2r} \quad \forall \Delta ABC, " = " \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$