

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$g_b g_c \leq r \left(r_a + 2a \cos \frac{A}{2} \right)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 g_a &\stackrel{\text{Bogdan Fustei}}{=} \sqrt{(s-a)^2 + \frac{4(s-a)(s-b)(s-c)}{a}} > s-a \text{ and analogs} \\
 \text{and so, } g_b g_c &> (s-b)(s-c) = rr_a \therefore g_b g_c \stackrel{?}{\leq} r \left(r_a + 2a \cos \frac{A}{2} \right) \\
 &\Leftrightarrow (g_b g_c - rr_a)^2 \stackrel{?}{\leq} 4r^2 a^2 \cdot \frac{s(s-a)}{bc} \stackrel{\text{via Bogdan Fustei}}{\Leftrightarrow} \\
 &\left((s-b)^2 + \frac{4(s-a)(s-b)(s-c)}{b} \right) \left((s-c)^2 + \frac{4(s-a)(s-b)(s-c)}{c} \right) + \\
 &\quad (s-b)^2 (s-c)^2 - \frac{4(s-a)^2 (s-b)(s-c)a^2}{bc} \stackrel{?}{\leq} 2g_b g_c (s-b)(s-c) \\
 \Leftrightarrow &\frac{(y(z+x) + 4zx)(z(x+y) + 4xy) + yz(z+x)(x+y) - 4x^2(y+z)^2}{(z+x)(x+y)} \stackrel{?}{\leq} 2g_b g_c \\
 &(x = s-a, y = s-b, z = s-c \Rightarrow a = y+z, b = z+x, c = x+y) \\
 \Leftrightarrow &\frac{2yz(5x^2 + 3xy + 3xz + yz)}{(z+x)(x+y)} \stackrel{?}{\leq} 2g_b g_c \stackrel{\text{via Bogdan Fustei}}{\Leftrightarrow} \\
 &\frac{y^2 z^2 (5x^2 + 3xy + 3xz + yz)^2}{(z+x)^2 (x+y)^2} \stackrel{?}{\leq} \frac{yz(y(z+x) + 4zx)(z(x+y) + 4xy)}{(z+x)(x+y)} \\
 \Leftrightarrow &4x^3 \left(x(y-z)^2 + y^3 + z^3 - yz(y+z) \right) \stackrel{?}{\geq} 0 \Leftrightarrow 4x^3(x+y+z)(y-z)^2 \stackrel{?}{\geq} 0 \\
 &\rightarrow \text{true} \because x, y, z > 0 \therefore g_b g_c \leq r \left(r_a + 2a \cos \frac{A}{2} \right) \forall \Delta ABC, \\
 &\quad \text{"=" iff } \Delta ABC \text{ is equilateral (QED)}
 \end{aligned}$$