ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$p_a \le \frac{\sqrt{9s^2 - 48Rr - 66r^2}}{9r} . h_a$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$p_a^2 \overset{?}{\le} \frac{9s^2 - 48Rr - 66r^2}{81r^2} . h_a^2 \overset{\text{via Bogdan Fustei and Mohamed Amine Ben Ajiba}}{\Leftrightarrow}$$

$$s(s-a) + \frac{s(3s+a)}{(2s+a)^2} . (b-c)^2 \overset{?}{\le}$$

$$\left(\frac{\frac{9s^3}{81(s-a)(s-b)(s-c)} - \frac{48abc}{81.4(s-a)(s-b)(s-c)}}{} - \frac{66}{81}\right)\left(\frac{4s(s-a)(s-b)(s-c)}{a^2}\right)$$

$$\Leftrightarrow \frac{x\big(y+z+2(x+y+z)\big)^2 + \big(y+z+3(x+y+z)\big)(y-z)^2}{\big(y+z+2(x+y+z)\big)^2}$$

$$\overset{?}{\le} \frac{12(x+y+z)^3 - 16(y+z)(z+x)(x+y) - 88xyz}{27(y+z)^2}$$

$$\left(x = s-a, y = s-b, z = s-c \Rightarrow a = y+z, b = z+x, c = x+y \text{ and } s = x+y+z\right)$$

$$\Leftrightarrow 12x^5 + 56x^4(y+z) + 80x^3(y+z)^2 - 88x^3.nx^2 +$$
$$36x^2\left((mx)^3 - 3nx^2(mx)\right) - 172x^2.nx^2.mx - 165x.nx^2.(mx)^2 +$$
$$72.nx^2.\left((mx)^3 - 3nx^2(mx)\right) + 216n^2x^4.mx \overset{?}{\ge} 0 \ \left(m = \frac{y+z}{x}, n = \frac{yz}{x^2}\right)$$

$$\Leftrightarrow n(72m^3 - 165m^2 - 280m - 88) + 36m^3 + 80m^2 + 56m + 12 \underset{(*)}{\overset{?}{\ge}} 0 \text{ and it's}$$

trivially true if : $72m^3 - 165m^2 - 280m - 88 \ge 0$ and when :

$72m^3 - 165m^2 - 280m - 88 < 0$, *then* : since $n \overset{AM-GM}{\le} \frac{m^2}{4}$

$\therefore$ in order to prove $(*)$, it suffices to prove :

$$\frac{m^2}{4} . (72m^3 - 165m^2 - 280m - 88) + 36m^3 + 80m^2 + 56m + 12 \overset{?}{\ge} 0$$

$$\Leftrightarrow (8m+3)(3m+2)^2(m-2)^2 \overset{?}{\ge} 0 \to \text{true} \because m > 0 \text{ as } x, y, z > 0 \Rightarrow (*) \text{ is true}$$

$$\therefore p_a \le \frac{\sqrt{9s^2 - 48Rr - 66r^2}}{9r} . h_a \text{ "=" iff } y = z \text{ and } y+z = 2x \Rightarrow$$

"=" iff Δ ABC is equilateral (QED)