ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$g_a \leq \sqrt{\frac{R+2r}{4r}}h_a$$

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$$g_a^2 \overset{?}{\leq} \left(\frac{R}{4r}+\frac{1}{2}\right).h_a^2 \overset{\text{via Bogdan Fustei}}{\Leftrightarrow} s(s-a)-\frac{s-a}{a}.(b-c)^2 \overset{?}{\leq}$$

$$\left(\frac{abc}{16(s-a)(s-b)(s-c)}+\frac{1}{2}\right)\left(s(s-a)-\frac{s(s-a)}{a^2}.(b-c)^2\right)$$

$$\Leftrightarrow \frac{sa-(b-c)^2}{a} \overset{?}{\leq} s.\frac{abc+8(s-a)(s-b)(s-c)}{16(s-a)(s-b)(s-c)}.\frac{4(s-b)(s-c)}{a^2}$$

$$\Leftrightarrow \big((y+z)(z+x)(x+y)+8xyz\big)(x+y+z) \overset{?}{\geq} 4(y+z)x\left(\begin{matrix}(y+z)(x+y+z)-\\(y-z)^2\end{matrix}\right)$$

$$\left(\begin{matrix}x=s-a, y=s-b, z=s-c \Rightarrow a=y+z, b=z+x, c=x+y \text{ and}\\ s=x+y+z\end{matrix}\right)$$

$$\Leftrightarrow x^3(y+z)-2x^2(y+z)^2+8x^2.yz+x\left((y+z)^3-3yz(y+z)\right)-$$

$$4x.yz(y+z)+yz(y+z)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow x^3(mx)-2x^2(mx)^2+8x^2.nx^2+x\left((mx)^3-3nx^2(mx)\right)-$$

$$4x.nx^2(mx)+nx^2(mx)^2 \overset{?}{\geq} 0 \left(m=\frac{y+z}{x}, n=\frac{yz}{x^2}\right)$$

$$\Leftrightarrow n(m^2-7m+8)+m(m-1)^2 \boxed{\overset{?}{\underset{(*)}{\geq}}} 0 \text{ and it's trivially true if :}$$

$$m^2-7m+8 \geq 0 \text{ and when : } m^2-7m+8<0, \text{ then : since } n \overset{\text{AM-GM}}{\leq} \frac{m^2}{4}$$

$$\therefore \text{ in order to prove } (*), \text{ it suffices to prove : } \frac{m^2}{4}.(m^2-7m+8)+m(m-1)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (m+1)(m-2)^2 \overset{?}{\geq} 0 \to \text{true} \because m>0 \text{ as } x,y,z>0 \Rightarrow (*) \text{ is true}$$

$$\therefore g_a \leq \sqrt{\frac{R+2r}{4r}}.h_a, ''='' \text{ iff } y=z \text{ and } y+z=2x \Rightarrow$$

$$''='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$