

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$3r \sqrt{\frac{3R}{2}} \leq w_a \cdot \sqrt{r_a} \leq m_a \cdot \sqrt{r_a} \leq \frac{s^2}{3\sqrt{3r}}$$

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$$\begin{aligned} m_a \cdot \sqrt{r_a} &\stackrel{?}{\leq} \frac{s^2}{3\sqrt{3r}} \stackrel{\text{squaring}}{\Leftrightarrow} \left(\frac{(b-c)^2 + 4s(s-a)}{4} \right) \cdot \frac{rs}{s-a} \stackrel{?}{\leq} \frac{s^4}{27r} \\ &\Leftrightarrow \frac{(b-c)^2 + 4s(s-a)}{4} \stackrel{?}{\leq} \frac{s^4(s-a)}{27(s-a)(s-b)(s-c)} \\ &\Leftrightarrow 4(x+y+z)^4 \stackrel{?}{\geq} 27yz \left((y-z)^2 + 4x(x+y+z) \right) \\ &\quad \left(x = s-a, y = s-b, z = s-c \Rightarrow a = y+z, b = z+x, c = x+y \right) \\ &\quad \text{and } s = x+y+z \\ &\Leftrightarrow 4x^4 + 16x^3(y+z) + 24x^2(y+z)^2 - 108x^2 \cdot yz + 16x(y+z)^3 - 108xyz(y+z) \\ &\quad + 4((y+z)^2 - 2yz)^2 + 70x^2y^2z^2 - 11yz((y+z)^2 - 2yz) \stackrel{?}{\geq} 0 \\ &\Leftrightarrow 4x^4 + 16x^3(mx) + 24x^2(mx)^2 - 108x^2 \cdot nx^2 + 16x(mx)^3 - 108x \cdot nx^2(mx) \\ &\quad + 4((mx)^2 - 2nx^2)^2 + 70x^2 \cdot n^2x^4 - 11nx^2((mx)^2 - 2nx^2) \stackrel{?}{\geq} 0 \\ &\quad \left(\frac{y+z}{x} = m \text{ and } \frac{yz}{x^2} = n \right) \Leftrightarrow 108n^2 - 27(m+2)^2 \cdot n + 4(m+1)^4 \stackrel{?}{\geq} 0 \end{aligned}$$

Now, LHS of ① is a quadratic polynomial in "n" with discriminant, $\delta = 729(m+2)^4 - (16)(108)(m+1)^4$

$$= -27(37m^4 + 40m^3 - 264m^2 - 608m - 368) \leq 0 \text{ if :}$$

$37m^4 + 40m^3 - 264m^2 - 608m - 368 \geq 0$ and then ① is definitely true

and when : $37m^4 + 40m^3 - 264m^2 - 608m - 368 < 0$, then : $\delta > 0$

and then, in order to prove ①, it suffices to prove : $216n \stackrel{?}{\leq} 27(m+2)^2 - \sqrt{\delta}$

and since $n \stackrel{\text{AM-GM}}{\leq} \frac{m^2}{4} \therefore$ it suffices to prove : $54m^2 \stackrel{?}{\leq} 27(m+2)^2 - \sqrt{\delta}$

$$\Leftrightarrow \sqrt{\delta} \stackrel{?}{\geq} 27(4 + 4m - m^2)$$

Now, $37m^4 + 40m^3 - 264m^2 - 608m - 368$

$$= (m^2 - 4m - 4)(37m^2 + 188m + 636) + 2688m + 2176$$

$$\Rightarrow m^2 - 4m - 4 = \frac{(37m^4 + 40m^3 - 264m^2 - 608m - 368) - 2688m - 2176}{37m^2 + 188m + 636}$$

$< 0 \Rightarrow 4 + 4m - m^2 > 0$ and so, following squaring, ② becomes :

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$$729(4 + 4m - m^2)^2 \stackrel{?}{\geq} -27(37m^4 + 40m^3 - 264m^2 - 608m - 368)$$

$$\Leftrightarrow \boxed{432(m+1)(4m+1)(m-2)^2 \stackrel{?}{\geq} 0} \rightarrow \text{true} \because m > 0 \Rightarrow \textcircled{2} \Rightarrow \textcircled{1} \text{ is true}$$

$$\therefore \boxed{m_a \cdot \sqrt{r_a} \leq \frac{s^2}{3\sqrt{3r}}} \text{ and again, } w_a \cdot \sqrt{r_a} \stackrel{?}{\geq} 3r \cdot \sqrt{\frac{3R}{2}} \stackrel{\text{squaring}}{\Leftrightarrow}$$

$$\left(s(s-a) \cdot \frac{4bc}{(b+c)^2} \right) \cdot \frac{rs}{s-a} \stackrel{?}{\geq} 9r^2 \cdot \frac{3abc}{8rs}$$

$$\Leftrightarrow 32(x+y+z)^3 \stackrel{?}{\geq} 27(y+z)(2x+y+z)^2$$

$$\Leftrightarrow 32x^3 - 12x^2(y+z) - 12x(y+z)^2 + 5(y+z)^3 \stackrel{?}{\geq} 0$$

$$\Leftrightarrow 32x^3 - 12x^2(mx) - 12x(mx)^2 + 5(mx)^3 \stackrel{?}{\geq} 0 \Leftrightarrow \boxed{(5m+8)(m-2)^2 \stackrel{?}{\geq} 0}$$

$$\rightarrow \text{true} \because m > 0 \therefore \boxed{w_a \cdot \sqrt{r_a} \geq 3r \cdot \sqrt{\frac{3R}{2}}} \text{ and } \because w_a \leq m_a, \text{ hence :}$$

$$3r \cdot \sqrt{\frac{3R}{2}} \leq w_a \cdot \sqrt{r_a} \leq m_a \cdot \sqrt{r_a} \leq \frac{s^2}{3\sqrt{3r}},$$

$$" = " \text{ for } m_a \cdot \sqrt{r_a} \leq \frac{s^2}{3\sqrt{3r}} \text{ iff } \Delta ABC \text{ is equilateral, " = " for } w_a \cdot \sqrt{r_a} \geq 3r \cdot \sqrt{\frac{3R}{2}}$$

$$\text{iff } b + c = 2a \text{ and " = " for } w_a \cdot \sqrt{r_a} \leq m_a \cdot \sqrt{r_a} \text{ iff } b = c \text{ (QED)}$$