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In any Δ ABC the following relationship holds :

$$\frac{r_b + r_c}{2w_a} + \frac{2w_a}{r_b + r_c} \le \frac{2s}{3\sqrt{3}r}$$

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Let $x = s - a, y = s - b, z = s - c$ & then : $a = y + z, b = z + x, c = x + y$

and $s = x + y + z$ and furthermore, we denote : $\frac{y+z}{x} = m$ and $\frac{yz}{x^2} = n$ and

then, we have the following set "S" of relations : $y^2 + z^2 = x^2(m^2 - 2n)$,

$y^3 + z^3 = x^3(m^3 - 3nn), y^4 + z^4 = x^4((m^2 - 2n)^2 - 2n^2)$,

$y^5 + z^5 = x^5\left(m((m^2 - 2n)^2 + n^2 - nm^2)\right), y^6 + z^6 = x^6((m^3 - 3nn)^2 - 2n^3)$,

$y^7 + z^7 = x^7\left(m((m^3 - 3nn)^2 - 2n^3) - mn((m^2 - 2n)^2 + n^2 - nm^2)\right)$,

$y^8 + z^8 = x^8(((m^2 - 2n)^2 - 2n^2)^2 - 2n^4)$ and now, $\frac{r_b + r_c}{2w_a} = \frac{\frac{rsa(s-a)}{(s-b)(s-c)(s-a)}}{\frac{4\sqrt{bc}}{b+c}.\sqrt{s(s-a)}}$

$$= \frac{rs.a(s-a)}{r^2s.\frac{4\sqrt{bc}}{b+c}.\sqrt{s(s-a)}} \Rightarrow \left(\frac{r_b + r_c}{2w_a}\right)^2 = \frac{a^2(s-a)^2}{r^2s.\frac{16bc}{(b+c)^2}.(s-a)}$$

$$= \frac{a^2(s-a)^2}{(s-a)(s-b)(s-c).\frac{16bc}{(b+c)^2}.(s-a)} \Rightarrow \left(\frac{r_b + r_c}{2w_a}\right)^2 = \frac{a^2(b+c)^2}{16bc(s-b)(s-c)}$$

and so, $\frac{r_b + r_c}{2w_a} + \frac{2w_a}{r_b + r_c} \overset{?}{\le} \frac{2s}{3\sqrt{3}r} \overset{\text{squaring}}{\Leftrightarrow} \left(\frac{r_b + r_c}{2w_a}\right)^2 + \left(\frac{2w_a}{r_b + r_c}\right)^2 + 2$

$$\overset{?}{\le} \frac{4s^2}{27r^2} = \frac{4s^3}{27(s-a)(s-b)(s-c)}$$

$$\Leftrightarrow \frac{a^2(b+c)^2}{16bc(s-b)(s-c)} + \frac{16bc(s-b)(s-c)}{a^2(b+c)^2} + 2 \overset{?}{\le} \frac{4s^3}{27(s-a)(s-b)(s-c)}$$

$$\Leftrightarrow \frac{(y+z)^2(2x+y+z)^2}{16yz(z+x)(x+y)} + \frac{16yz(z+x)(x+y)}{(y+z)^2(2x+y+z)^2} + 2 \overset{?}{\le} \frac{4(x+y+z)^3}{27xyz}$$

$\Leftrightarrow 256x^7(y+z)^2 + 1280x^6(y+z)^3 + 2192x^5(y^2+z^2)^2 - 4544x^5y^2z^2 +$

$5568x^5yz(y^2+z^2) + 1952x^4(y^5+z^5) + 3872x^4yz(y^3+z^3) -$

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$$11968x^4y^2z^2(y+z)+1016x^3(y^6+z^6)+3376x^3yz(y^4+z^4)-$$
$$6008x^3y^2z^2(y^2+z^2)-30560x^3y^3z^3+296x^2(y^7+z^7)+2424x^2yz(y^5+z^5)+$$
$$4520x^2y^2z^2(y^3+z^3)-10312x^2y^3z^3(y+z)+37x(y^8+z^8)+744xyz(y^6+z^6)$$
$$+2860xy^2z^2(y^4+z^4)+5336xy^3z^3(y^2+z^2)-546xy^4z^4+64yz(y+z)^7z^4 \overset{?}{\underset{①}{\geq}} 0$$

& via set of relations "S", to prove ①, suffices to prove, following simplification :

$$\overbrace{(6912+13824m+10368m^2+3456m^3+864m^4)}^{\Omega_1}n^2-$$
$$\overbrace{(64m^7+448m^6+352m^5-2720m^4-5888m^3-3200m^2)}^{\Omega_2}n-$$
$$\overbrace{(37m^8+296m^7+1016m^6+1952m^5+2192m^4+1280m^3+256m^2)}^{\Omega_3}+$$
$$6912n^4+13824(m+1)n^3 \overset{?}{\underset{②}{\leq}} 0 \text{ and } \because 6912n^4+13824(m+1)n^3$$
$$\overset{AM-GM}{\leq} 6912n^2\left(\frac{m^4}{16}\right)+13824(m+1)n^2\left(\frac{m^2}{4}\right) \therefore \text{in order to prove ②,}$$

it suffices to prove : $\Omega_1.n^2-\Omega_2.n-\Omega_3+432n^2m^4+3456(m+1)m^2n^2 \overset{?}{\leq} 0$

$$\Leftrightarrow \overbrace{(6912+13824m+13824m^2+6912m^3+1296m^4)}^{\sigma_1}n^2-$$
$$\overbrace{(64m^7+448m^6+352m^5-2720m^4-5888m^3-3200m^2)}^{\sigma_2}n-$$
$$\overbrace{(37m^8+296m^7+1016m^6+1952m^5+2192m^4+1280m^3+256m^2)}^{\sigma_3} \overset{?}{\underset{③}{\leq}} 0$$

Now, LHS of ③ is a quadratic polynomial in "n" with discriminant,

$\boxed{\delta}=\sigma_2^2+4\sigma_1\sigma_3 \boxed{>0}$ ($\because \sigma_1,\sigma_3>0$) and so, in order to prove ③,

it suffices to prove : $2\sigma_1.n \overset{?}{\underset{(m)}{\leq}} \sigma_2+\sqrt{\delta}$ AND $2\sigma_1.n \overset{?}{\underset{(n)}{\geq}} \sigma_2-\sqrt{\delta}$

Now, since $n \overset{AM-GM}{\leq} \frac{m^2}{4} \therefore$ in order to prove (m), it suffices to prove :

$$\sigma_1 . \frac{m^2}{2} \overset{?}{\leq} \sigma_2 + \sqrt{\delta} \Leftrightarrow -8m^2(m+2)^2(8m^3 - 57m^2 - 192m - 208) \overset{?}{\leq} \sqrt{\delta}$$

and it's trivi$ally$ true if : $8m^3 - 57m^2 - 192m - 208 \geq 0$ and when :

$8m^3 - 57m^2 - 192m - 208 < 0$, then it suffices to prove :

$$64m^4(m+2)^4(8m^3 - 57m^2 - 192m - 208)^2 \overset{?}{\leq} \sigma_2^2 + 4\sigma_1\sigma_3$$

$$\Leftrightarrow \boxed{6912m^2(4m+1)(3m^2+4m+4)(m+2)^6(m-2)^2 \overset{?}{\geq} 0} \to \text{true} \because m > 0$$

$\Rightarrow$ (m) is true and also, $2\sigma_1 . n + \sqrt{\delta} > \sqrt{\delta} = \sqrt{\sigma_2^2 + 4\sigma_1\sigma_3} > \sqrt{\sigma_2^2} = |\sigma_2| \geq \sigma_2$

$\Rightarrow 2\sigma_1 . n > \sigma_2 - \sqrt{\delta} \Rightarrow$ (n) is true (strict inequality) $\therefore$ (m) and (n) are both true

$$\Rightarrow ③ \Rightarrow ② \Rightarrow ① \text{ is true} \therefore \frac{r_b + r_c}{2w_a} + \frac{2w_a}{r_b + r_c} \leq \frac{2s}{3\sqrt{3}r} \ \forall \Delta ABC,$$

" =" iff Δ ABC is equilateral (QED)