ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sqrt{\frac{r_b + r_c}{2r_a}} + \sqrt{\frac{2r_a}{r_b + r_c}} \le \sqrt{\frac{2R}{r}}$$

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$$\frac{r_b + r_c}{2r_a} = \frac{4R\cos^2\frac{A}{2}}{2.4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.\tan\frac{A}{2}} = \frac{1 - S^2}{S(C + S)}$$

$$\left(C = \cos\frac{B - C}{2} \text{ and } S = \sin\frac{A}{2}\right) \therefore \frac{r_b + r_c}{2r_a} \overset{①}{=} \frac{1 - S^2}{S(C + S)} \text{ and } \frac{2R}{r} = \frac{2R}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$= \frac{1}{S(C - S)} \therefore \frac{2R}{r} \overset{②}{=} \frac{1}{S(C - S)} \text{ and now, } \sqrt{\frac{r_b + r_c}{2r_a}} + \sqrt{\frac{2r_a}{r_b + r_c}} \overset{?}{\le} \sqrt{\frac{2R}{r}}$$

$$\Leftrightarrow \frac{r_b + r_c}{2r_a} + \frac{2r_a}{r_b + r_c} + 2 \overset{?}{\le} \frac{2R}{r} \overset{\text{via ① and ②}}{\Leftrightarrow} \frac{1 - S^2}{S(C + S)} + \frac{S(C + S)}{1 - S^2} + 2 \overset{?}{\le} \frac{1}{S(C - S)}$$

$$\Leftrightarrow 2 - 2C^2 - S^2 + C^2S^2 + CS - C^3S \overset{?}{\ge} 0$$

$$\Leftrightarrow 2(1 - C^2) - S^2(1 - C^2) + CS(1 - C^2) \overset{?}{\ge} 0 \Leftrightarrow (1 - C^2)\big(2 + S(C - S)\big) \overset{?}{\ge} 0$$

$$\rightarrow \text{true} \because C^2 = \cos^2\frac{B - C}{2} \le 1 \text{ and } C - S = \frac{b + c}{a}.\sin\frac{A}{2} - \sin\frac{A}{2} > \sin\frac{A}{2} - \sin\frac{A}{2} = 0$$

$$\therefore \sqrt{\frac{r_b + r_c}{2r_a}} + \sqrt{\frac{2r_a}{r_b + r_c}} \le \sqrt{\frac{2R}{r}} \ \forall \Delta ABC, '' ='' \text{ iff } b = c \text{ (QED)}$$