ROMANIAN MATHEMATICAL MAGAZINE

In $\Delta ABC$ the following relationship holds:

$$max\left\{\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}},\frac{a+b}{a+c}+\frac{a+c}{a+b}\right\}\le\sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}}\le\frac{w_b}{w_c}+\frac{w_c}{w_b}$$

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*Here, WLOG assume that* $b\ge c\Rightarrow B\ge C\Longrightarrow All\left\{\sqrt{\frac{b}{c}},\frac{a+b}{a+c},\sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}},\frac{w_c}{w_b}\right\}\ge 1$

*The function* $f(x)=x+\frac{1}{x}$ *is strictly increasing for* $x\ge 1$.

$\therefore$ *In order to prove the original inequality, it suffices to prove that:*

$$max\left\{\sqrt{\frac{b}{c}},\frac{a+b}{a+c}\right\}\le\sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}\le\frac{w_c}{w_b}$$

*Via Sine Law:* $\frac{\sin B}{\sin C}=\frac{b}{c}\Rightarrow\frac{2\sin\frac{B}{2}\cos\frac{B}{2}}{2\sin\frac{C}{2}\cos\frac{C}{2}}=\frac{b}{c}\Rightarrow\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}=\frac{b}{c}\cdot\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}$

*Since* $B\ge C\Rightarrow\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}\ge 1,\Leftrightarrow\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}\ge\frac{b}{c}\cdot 1\Leftrightarrow\sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}\overset{①}{\ge}\sqrt{\frac{b}{c}}$

*Now, Via HAS Formula:* $\sin^2\frac{B}{2}=\frac{(s-a)(s-c)}{ac}\Longrightarrow\frac{\sin^2\frac{B}{2}}{\sin^2\frac{C}{2}}$

$$=\frac{b(a+b-c)}{c(a+c-b)}\overset{?}{\ge}\left(\frac{a+b}{a+c}\right)^4$$

$$\Leftrightarrow(b-c)\underbrace{\left[a^5+a^4(b+c)+2a^3bc+2a^2bc(b+c)+abc(3b^2-bc+3c^2)+bc(b^3+c^3)\right]}_{>0}\overset{?}{\ge}0$$

*which is trivially true* $\because b\ge c\quad\Longrightarrow\sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}\overset{②}{\ge}\frac{a+b}{a+c}$

$$Finally, \frac{w_c}{w_b} = \frac{b(a+c)\cos\frac{C}{2}}{c(a+b)\cos\frac{B}{2}} = \left(\frac{b\cdot\cos\frac{C}{2}}{c\cdot\cos\frac{B}{2}}\right)\cdot\frac{a+c}{a+b} = \left(\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}\right)\cdot\frac{a+c}{a+b} \overset{Via\ (2)}{\geq} \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}$$

$\mapsto (3)$

$$\therefore\ Via\ (1),(2)\ and\ (3):\ max\left\{\sqrt{\frac{b}{c}},\frac{a+b}{a+c}\right\} \leq \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}} \leq \frac{w_c}{w_b}.\ Which\ follows:$$

$$max\left\{\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}},\frac{a+b}{a+c}+\frac{a+c}{a+b}\right\} \leq \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}} + \sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}} \leq \frac{w_b}{w_c}+\frac{w_c}{w_b}$$

*Equality holds if the triangle is isosceles* $(b = c)$.