ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$max\left\{\frac{h_b+h_c}{2r_a}+\frac{2r_a}{h_b+h_c},\frac{r_b+r_c}{2h_a}+\frac{2h_a}{r_b+r_c}\right\}\le\frac{R}{r}$$

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$$\frac{h_b+h_c}{2r_a}+\frac{2r_a}{h_b+h_c}\overset{?}{\le}\frac{R}{r}\Leftrightarrow\frac{\frac{1}{b}+\frac{1}{c}}{\frac{1}{s-a}}+\frac{\frac{1}{s-a}}{\frac{1}{b}+\frac{1}{c}}\overset{?}{\le}\frac{abc}{4(s-a)(s-b)(s-c)}$$
$$\Leftrightarrow\frac{b^2c^2+(s-a)^2(b+c)^2}{bc(s-a)(b+c)}\overset{?}{\le}\frac{abc}{4(s-a)(s-b)(s-c)}\Leftrightarrow$$
$$(z+x)^2(x+y)^2(y+z)(2x+y+z)\overset{?}{\ge}4yz((z+x)^2(x+y)^2+x^2(2x+y+z)^2)$$
$$(x=s-a,y=s-b,z=s-c\Rightarrow a=y+z,b=z+x,c=x+y)$$
$$\Leftrightarrow 2x^5(y+z)+5x^4(y+z)^2-20x^4yz+4x^3(y+z)^3-20x^3yz(y+z)+$$
$$x^2((y+z)^2-2yz)^2-8x^2y^2z^2+2x^2yz((y+z)^2-2yz)+$$
$$2xyz\left((y+z)^3-3yz(y+z)\right)+y^2z^2(y-z)^2\overset{?}{\ge}0\text{ and }\because y^2z^2(y-z)^2\ge0$$
$$\therefore\text{ it suffices to prove : }2m+5m^2-20n+4m^3-20mn+(m^2-2n)^2-8n^2+$$
$$2n(m^2-2n)+2n(m^3-3mn)\overset{?}{\ge}0\left(m=\frac{y+z}{x},n=\frac{yz}{x^2}\right)$$
$$\Leftrightarrow(6m+8)n^2-(2m^3-2m^2-20m-20)n-(m^4+4m^3+5m^2+2m)\underset{(*)}{\overset{?}{\le}}0$$

Now, LHS of $(*)$ is *a* quadratic polynomial in "n" with discriminant, $\delta=$
$$(2m^3-2m^2-20m-20)^2+4(6m+8)(m^4+4m^3+5m^2+2m)$$
$$=4\left(m^6+4m^5+13m^4+62m^3+172m^2+216m+100\right)>0\text{ and so,}$$
in order to prove $(*)$, it suffices to prove :

$$2(6m+8)n\underset{①}{\overset{?}{\le}}2m^3-2m^2-20m-20+\sqrt{\delta}\text{ AND}$$

$$2(6m+8)n\underset{②}{\overset{?}{\ge}}2m^3-2m^2-20m-20-\sqrt{\delta}$$

Now, since $n\overset{AM-GM}{\le}\frac{m^2}{4}\therefore$ in order to prove ①, it suffices to prove :

$$(3m+4)m^2\overset{?}{\le}2m^3-2m^2-20m-20+\sqrt{\delta}\Leftrightarrow m^3+6m^2+20m+20\overset{?}{\le}\sqrt{\delta}$$
$$\Leftrightarrow4\left(m^6+4m^5+13m^4+62m^3+172m^2+216m+100\right)\overset{?}{\ge}$$
$$(m^3+6m^2+20m+20)^2\Leftrightarrow\boxed{m(3m+4)(m+2)^2(m-2)^2\overset{?}{\ge}0}\to\text{true}\because m>0$$
$$\Rightarrow ①\text{ is true and also, }2(6m+8)n+\sqrt{\delta}>\sqrt{\delta}$$
$$=\sqrt{(2m^3-2m^2-20m-20)^2+4(6m+8)(m^4+4m^3+5m^2+2m)}>$$
$$\sqrt{(2m^3-2m^2-20m-20)^2}=\left|2m^3-2m^2-20m-20\right|\ge$$

$$2m^3 - 2m^2 - 20m - 20 \Rightarrow 2(6m+8)n > 2m^3 - 2m^2 - 20m - 20 - \sqrt{\delta}$$

$\Rightarrow$ ② is true (strict inequality) $\therefore$ ① and ② are both true $\Rightarrow (*)$ is true

$$\therefore \frac{h_b + h_c}{2r_a} + \frac{2r_a}{h_b + h_c} \le \frac{R}{r} \text{ and again, } \frac{r_b + r_c}{2h_a} + \frac{2h_a}{r_b + r_c} \overset{?}{\le} \frac{R}{r}$$

$$\Leftrightarrow \frac{\frac{1}{s-b} + \frac{1}{s-c}}{\frac{4}{a}} + \frac{\frac{4}{a}}{\frac{1}{s-b} + \frac{1}{s-c}} \overset{?}{\le} \frac{abc}{4(s-a)(s-b)(s-c)}$$

$$\Leftrightarrow \frac{a^4 + 16(s-b)^2(s-c)^2}{4a^2(s-b)(s-c)} \overset{?}{\le} \frac{abc}{4(s-a)(s-b)(s-c)}$$

$$\Leftrightarrow (y+z)^3(x^2 + x(y+z) + yz) \overset{?}{\ge} x((y+z)^4 + 16y^2z^2)$$

$$\Leftrightarrow (mx)^3(x^2 + x(mx) + nx^2) \overset{?}{\ge} x((mx)^4 + 16n^2x^4) \Leftrightarrow m^3(n+1) \underset{(**)}{\overset{?}{\ge}} 16n^2$$

$$\text{Now, via AM} - \text{GM}, m^2 \ge 4n \therefore m^3(n+1) \ge 4n(2\sqrt{n})(2\sqrt{n}) = 16n^2$$

$$\Rightarrow (**) \text{ is true} \therefore \frac{r_b + r_c}{2h_a} + \frac{2h_a}{r_b + r_c} \le \frac{R}{r} \text{ and so,}$$

$$max\left\{\frac{h_b + h_c}{2r_a} + \frac{2r_a}{h_b + h_c}, \frac{r_b + r_c}{2h_a} + \frac{2h_a}{r_b + r_c}\right\} \le \frac{R}{r} \ \forall \Delta \ ABC,$$

'' ='' iff Δ ABC is equilateral (QED)