ROMANIAN MATHEMATICAL MAGAZINE

If in $\Delta ABC, a = \max\{a,b,c\}$ then:

$$\frac{b+c}{2a}+\frac{2a}{b+c}\le\frac{2s^2}{9Rr}-1$$

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$$\frac{2s^2}{9Rr}=\frac{2s^3}{9Rrs}=\frac{8s^3}{9(4Rrs)}=\frac{(a+b+c)^3}{9abc}$$

$$Let\ x=\frac{b+c}{a}>1\ (as\ in\ \Delta ABC\ b+c>a)$$

$$x=\frac{b+c}{a}=\frac{b+c}{\sqrt{a^2}}\overset{a=\max\{a,b,c\}}{\le}\frac{b+c}{\sqrt{bc}}\overset{AM-GM}{\le}\frac{b+c}{\frac{b+c}{2}}=2\ \ so\ 1<x\le 2$$

$$bc\overset{AM-GM}{\le}\left(\frac{b+c}{2}\right)^2\ \ or\ \ bc\overset{x=\frac{b+c}{a}}{\le}\frac{a^2x^2}{4}\ \&\ abc\le\frac{a^3x^2}{4}$$

$$\frac{(a+b+c)^3}{9abc}=\frac{a^3\left(1+\frac{b+c}{a}\right)}{9abc}\ge\frac{a^3(1+x)^3}{9.\frac{a^3x^2}{4}}=\frac{4(1+x)^3}{9x^2}$$

$$We\ need\ to\ show:$$

$$\frac{b+c}{2a}+\frac{2a}{b+c}\le\frac{2s^2}{9Rr}-1\ \ or\ \ \frac{2s^2}{9Rr}-\left(\frac{b+c}{2a}+\frac{2a}{b+c}\right)\ge 1\ or$$

$$\frac{2s^2}{9Rr}-\left(\frac{b+c}{2a}+\frac{2a}{b+c}\right)-1\ge 0\ or\ \frac{4(1+x)^3}{9x^2}-\frac{x}{2}-\frac{2}{x}-1\ge 0$$

$$L.H.S=\frac{4(1+x)^3}{9x^2}-\frac{x}{2}-\frac{2}{x}-1=\frac{8(1+x)^3-9x^3-36x-18x^2}{18x^2}=$$

$$=\frac{8-12x+6x^2-x^3}{18x^2}=\frac{(2-x)^3}{18x^2}\ge 0\ true\ as\ 1<x\le 2$$

Equality holds for an equilateral triangle.