ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with $\omega \to$ Brocard's angle, the following relationship holds :

$$\frac{b+c}{2a}+\frac{2a}{b+c} \leq \min\left\{\frac{1}{\sin\omega}, \frac{s^2}{4Rr}-\frac{11}{8}, \frac{s^2}{27r^2}+1\right\}$$

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Let $x = s - a, y = s - b, z = s - c$; then : $a = y + z, b = z + x, c = x + y$ and $s = x + y + z$ and furthermore, we denote : $\frac{y+z}{x} = m$ and $\frac{yz}{x^2} = n$ and then, we have the following set S of relations : $y^2 + z^2 = x^2(m^2 - 2n), y^3 + z^3 = x^3(m^3 - 3nn), y^4 + z^4 = x^4((m^2 - 2n)^2 - 2n^2), y^5 + z^5 = x^5\left(m((m^2 - 2n)^2 + n^2 - nm^2)\right), y^6 + z^6 = x^6((m^3 - 3nn)^2 - 2n^3), y^7 + z^7 = x^7\left(m((m^3 - 3nn)^2 - 2n^3) - mn((m^2 - 2n)^2 + n^2 - nm^2)\right)$,

$y^8 + z^8 = x^8(((m^2 - 2n)^2 - 2n^2)^2 - 2n^4)$ and now, $\frac{b+c}{2a}+\frac{2a}{b+c} \overset{?}{\leq} \frac{1}{\sin\omega}$

$$\Leftrightarrow \frac{4\sum_{cyc} a^2b^2}{2\sum_{cyc} a^2b^2 - \sum_{cyc} a^4} - 2 \overset{?}{\geq} \frac{(b+c)^4 + 16a^4}{4a^2(b+c)^2}$$

$$\Leftrightarrow \frac{(y+z)^4 + (z+x)^4 + (x+y)^4}{2xyz(x+y+z)} \overset{?}{\geq} \frac{(2x+y+z)^4 + 16(y+z)^4}{(y+z)^2(2x+y+z)^2}$$

$\Leftrightarrow 4x^6(y-z)^2 + 12x^5(y-z)^2(y+z) + 21x^4(y^2+z^2)^2 - 76x^4y^2z^2 + 4x^4yz(y^2+z^2) + 22x^3(y^5+z^5) + 30x^3yz(y^3+z^3) - 20x^3y^2z^2(y+z) + 15x^2(y^6+z^6) + 27x^2yz(y^4+z^4) - 23x^2y^2z^2(y+z)^2 - 24y^3z^3 + 6x(y^7+z^7) + 11xyz(y^5+z^5) - 25xy^2z^2(y^3+z^3) - 88xy^3z^3(y+z) + y^8 + z^8 + 6yz(y^6+z^6) + 17y^2z^2(y^4+z^4) + 30y^3z^3(y^2+z^2) + 36y^4z^4 \underset{①}{\overset{?}{\geq}} 0$ and $\because 4x^6(y-z)^2 + 12x^5(y-z)^2(y+z) \geq 0$ & via set of relations "S", to prove ①, suffices to prove, following simplification : $(m+2)^2n^2 - (2m^4 + 31m^3 + 63m^2 + 80m + 80)n + m^2(m^4 + 6m^3 + 15m^2 + 22m + 21) \underset{②}{\overset{?}{\geq}} 0$ and discriminant δ of LHS of ② =

$$(2m^4 + 31m^3 + 63m^2 + 80m + 80)^2 - 4m^2(m+2)^2\begin{pmatrix}m^4 + 6m^3 + 15m^2 + \\ 22m + 21\end{pmatrix}$$

$= 84m^7 + 1041m^6 + 3802m^5 + 8573m^4 + 14352m^3 + 16144m^2 + 12800m + 6400 > 0 \therefore$ in order to prove ②, it suffices to prove :

$2(m+2)^2n \overset{?}{\leq} (2m^4 + 31m^3 + 63m^2 + 80m + 80) - \sqrt{\delta}$ and $\because n \overset{AM-GM}{\leq} \frac{m^2}{4}$

$\therefore$ it suffices to prove : $2\sqrt{\delta} \overset{?}{\leq} 2(2m^4 + 31m^3 + 63m^2 + 80m + 80) - m^2(m+2)^2$

$$\Leftrightarrow 2\sqrt{\delta} \overset{?}{\leq} 3m^4 + 58m^3 + 122m^2 + 160m + 160 \overset{\text{squaring}}{\Leftrightarrow}$$

$$\boxed{m^2(m+2)^2(3m+2)^2(m-2)^2 \overset{?}{\geq} 0} \rightarrow \text{true} \Rightarrow \text{① is true} \therefore \frac{b+c}{2a} + \frac{2a}{b+c} \leq \frac{1}{\sin\omega}$$

$$\text{Again}, \frac{b+c}{2a} + \frac{2a}{b+c} \overset{?}{\leq} \frac{s^2}{4Rr} - \frac{11}{8} \overset{\text{via earlier substitutions}}{\Leftrightarrow}$$

$$\left(8(x+y+z)^3 - 11(y+z)(z+x)(x+y)\right)(2x+y+z) \overset{?}{\geq}$$

$$4(z+x)(x+y)\left((2x+y+z)^2 + 4(y+z)^2\right) \Leftrightarrow 2x^3(y+z) + 3x^2(y+z)^2 -$$

$$16x^2yz + 9x(y^3+z^3) - 11xyz(y+z) + 8(y^2+z^2)^2 - 30y^2z^2 + yz(y^2+z^2) \overset{?}{\geq} 0$$

$$\overset{\text{via set of relations S and following simplification}}{\Leftrightarrow} 8m^4 + 9m^3 + 3m^2 + 2m - n(31m^2 + 38m + 16) \overset{?}{\geq} 0 \text{ and}$$

$$\because n \overset{\text{AM-GM}}{\leq} \frac{m^2}{4} \therefore \text{it suffices to prove :}$$

$$4(8m^4 + 9m^3 + 3m^2 + 2m) - m^2(31m^2 + 38m + 16) \overset{?}{\geq} 0$$

$$\Leftrightarrow \boxed{m(m+2)(m-2)^2 \overset{?}{\geq} 0} \rightarrow \text{true} \therefore \frac{b+c}{2a} + \frac{2a}{b+c} \leq \frac{s^2}{4Rr} - \frac{11}{8}$$

$$\text{Finally}, \frac{b+c}{2a} + \frac{2a}{b+c} \overset{?}{\leq} \frac{s^2}{27r^2} + 1 \overset{\text{via earlier substitutions}}{\Leftrightarrow}$$

$$2(y+z)(2x+y+z)\left((x+y+z)^3 + 27xyz\right) \overset{?}{\geq} 27xyz\left((2x+y+z)^2 + 4(y+z)^2\right)$$

$$\Leftrightarrow 4x^4(y+z) + 14x^3(y+z)^2 - 108x^3yz + 18x^2(y+z)^3 + 10x(y^2+z^2)^2 -$$

$$122xy^2z^2 - 41xyz(y^2+z^2) + 2(y+z)^5 \overset{?}{\geq} 0 \overset{\text{via set of relations S and following simplification}}{\Leftrightarrow}$$

$$2m^5 + 10m^4 + 18m^3 + 14m^2 + 4m \overset{?}{\geq} (81m^2 + 108)n \text{ and} \because n \overset{\text{AM-GM}}{\leq} \frac{m^2}{4}$$

$$\therefore \text{it suffices to prove : } 4\left(2m^5 + 10m^4 + 18m^3 + 14m^2 + 4m\right) \overset{?}{\geq} m^2(81m^2 + 108)$$

$$\Leftrightarrow \boxed{m(8m^2 - 9m + 4)(m-2)^2 \overset{?}{\geq} 0} \rightarrow \text{true} \because \text{discriminant of } 8m^2 - 9m + 4 =$$

$$81 - 128 < 0 \Rightarrow 8m^2 - 9m + 4 > 0 \therefore \frac{b+c}{2a} + \frac{2a}{b+c} \leq \frac{s^2}{27r^2} + 1 \therefore$$

$$\frac{b+c}{2a} + \frac{2a}{b+c} \leq \min\left\{\frac{1}{\sin\omega}, \frac{s^2}{4Rr} - \frac{11}{8}, \frac{s^2}{27r^2} + 1\right\} \forall \Delta ABC,$$

$'' ='' $ iff Δ ABC is equilateral (QED)$'' ='' $ iff Δ ABC is equilateral (QED)