ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{w_a}{h_a}+\frac{h_a}{w_a}\le\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}\le\min\left\{\frac{w_b}{w_c}+\frac{w_c}{w_b},\frac{2w_a}{h_a}\right\}$$

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$$\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}=\frac{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}-1+\sin^2\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$=\frac{2+\frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{2R}-1+\sin^2\frac{A}{2}}{\frac{1}{2}(C+S)}\left(C=\cos\frac{B-C}{2},S=\sin\frac{A}{2}\right)=\frac{1+S^2+S(C-S)}{\frac{1}{2}(C+S)}$$

$$\therefore\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}=\frac{2(1+SC)}{C+S}\to(1)\text{ and so, via }(1),\frac{w_a}{h_a}+\frac{h_a}{w_a}\overset{?}{\le}\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}$$

$$\Leftrightarrow C+\frac{1}{C}\overset{?}{\le}\frac{2(1+SC)}{C+S}\Leftrightarrow 2C+2SC^2\overset{?}{\ge}C+S+C^3+SC^2\Leftrightarrow C-C^3\overset{?}{\ge}S-SC^2$$

$$\Leftrightarrow(1-C^2)(C-S)\overset{?}{\ge}0\to\text{true}\because C^2=\cos^2\frac{B-C}{2}\le1\text{ and }\because C=\cos\frac{B-C}{2}=$$

$$\frac{b+c}{a}.\sin\frac{A}{2}\overset{b+c>a}{>}\sin\frac{A}{2}=S\therefore\boxed{\frac{w_a}{h_a}+\frac{h_a}{w_a}\le\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}}\text{ and again,}$$

$$\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}\overset{?}{\le}\frac{w_b}{w_c}+\frac{w_c}{w_b}$$

$$\Leftrightarrow\frac{\frac{s(s-b)}{ca}+\frac{\cos^2\frac{C}{2}}{ab}}{\cos\frac{B}{2}\cos\frac{C}{2}}\overset{?}{\le}\frac{s(s-b)-\frac{s(s-b)(c-a)^2}{(c+a)^2}+s(s-c)-\frac{s(s-c)(a-b)^2}{(a+b)^2}}{\frac{2ca}{c+a}.\cos\frac{B}{2}.\frac{2ab}{a+b}.\cos\frac{C}{2}}$$

$$\Leftrightarrow 4as\Big(s(b+c)-(b^2+c^2)\Big)(a+b)(c+a)\overset{?}{\le}$$

$$sa(a+b)^2(c+a)^2-s(s-b)(c-a)^2(a+b)^2-s(s-c)(a-b)^2(c+a)^2$$

$$\Leftrightarrow 4(y+z)\begin{pmatrix}(x+y+z)(2x+y+z)-\\((z+x)^2+(x+y)^2)\end{pmatrix}(2z+x+y)(2y+z+x)\overset{?}{\le}$$

$$(y+z)(2z+x+y)^2(2y+z+x)^2-y(z-x)^2(2z+x+y)^2-$$

$$z(x-y)^2(2y+z+x)^2\begin{pmatrix}x=s-a,y=s-b,z=s-c\Rightarrow\\a=y+z,b=z+x,c=x+y,s=x+y+z\end{pmatrix}$$

$$\Leftrightarrow x(y^2-z^2)^2+y^5+z^5+y^4z+yz^4\overset{?}{\ge}2y^3z^2+2y^2z^3\to\text{true}$$

$$\because y^5+yz^4\overset{AM-GM}{\ge}2y^3z^2\text{ and }z^5+y^4z\overset{AM-GM}{\ge}2y^2z^3$$

$$\therefore \boxed{\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}} + \frac{\cos\frac{C}{2}}{\cos\frac{B}{2}} \leq \frac{w_b}{w_c} + \frac{w_c}{w_b}} \text{ and finally, via } (1), \frac{\cos\frac{B}{2}}{\cos\frac{C}{2}} + \frac{\cos\frac{C}{2}}{\cos\frac{B}{2}} \overset{?}{\leq} \frac{2w_a}{h_a}$$

$$\Leftrightarrow \frac{2(1+SC)}{C+S} \overset{?}{\leq} \frac{2}{C} \Leftrightarrow SC^2 \overset{?}{\leq} S \to \text{true} \because C^2 = \cos^2\frac{B-C}{2} \leq 1$$

$$\therefore \boxed{\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}} + \frac{\cos\frac{C}{2}}{\cos\frac{B}{2}} \leq \frac{2w_a}{h_a}} \text{ and so, } \frac{w_a}{h_a} + \frac{h_a}{w_a} \leq \frac{\cos\frac{B}{2}}{\cos\frac{C}{2}} + \frac{\cos\frac{C}{2}}{\cos\frac{B}{2}} \leq$$

$$\min\left\{\frac{w_b}{w_c} + \frac{w_c}{w_b}, 2\frac{w_a}{h_a}\right\} \forall \Delta ABC, '' ='' \text{ iff } b = c \text{ (QED)}$$