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In any Δ ABC the following relationship holds :

$$max\left\{\frac{2w_a}{h_a}, \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}}\right\} \le \sqrt{\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}}+\sqrt{\frac{\tan\frac{C}{2}}{\tan\frac{B}{2}}} \le \frac{r_b+r_c}{w_a}$$

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$$\frac{2w_a}{h_a} \boxed{\le} \frac{2.\sqrt{s(s-a)}.a}{2.\sqrt{s(s-a)(s-b)(s-c)}} = \frac{s-b+s-c}{\sqrt{(s-b)(s-c)}} = \sqrt{\frac{s-b}{s-c}}+\sqrt{\frac{s-c}{s-b}}$$

$$= \sqrt{\frac{r_c}{r_b}}+\sqrt{\frac{r_b}{r_c}} = \sqrt{\frac{\tan\frac{C}{2}}{\tan\frac{B}{2}}}+\sqrt{\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}} \text{ and again, } \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}} \overset{?}{\le} \sqrt{\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}}+\sqrt{\frac{\tan\frac{C}{2}}{\tan\frac{B}{2}}}$$

$$\overset{\text{squaring}}{\Leftrightarrow} \frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}+\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}} \overset{?}{\le} \frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}+\frac{\tan\frac{C}{2}}{\tan\frac{B}{2}} \Leftrightarrow \frac{\sqrt{\frac{(s-c)(s-a)}{ca}}}{\sqrt{\frac{(s-a)(s-b)}{ab}}}+\frac{\sqrt{\frac{(s-a)(s-b)}{ab}}}{\sqrt{\frac{(s-c)(s-a)}{ca}}} \overset{?}{\le} \frac{s-c}{s-b}+\frac{s-b}{s-c}$$

$$\Leftrightarrow \sqrt{\frac{b}{c}.\frac{s-c}{s-b}}+\sqrt{\frac{c}{b}.\frac{s-b}{s-c}} \underset{(*)}{\overset{?}{\le}} \frac{s-c}{s-b}+\frac{s-b}{s-c}$$

$$\text{Now, LHS of } (*) \overset{CBS}{\le} \sqrt{\frac{b}{c}+\frac{c}{b}}.\sqrt{\frac{s-c}{s-b}+\frac{s-b}{s-c}} \overset{?}{\le} \frac{s-c}{s-b}+\frac{s-b}{s-c}$$

$$\Leftrightarrow \frac{b}{c}+\frac{c}{b}-2 \overset{?}{\le} \frac{s-c}{s-b}-1+\frac{s-b}{s-c}-1 \Leftrightarrow \frac{(b-c)^2}{bc} \overset{?}{\le} \frac{b-c}{s-b}-\frac{b-c}{s-c}$$

$$\Leftrightarrow \frac{(b-c)^2}{bc} \overset{?}{\le} \frac{(b-c)^2}{(s-b)(s-c)} \Leftrightarrow \frac{(b-c)^2}{bc(s-b)(s-c)}.(-s^2+sa+bc-bc) \overset{?}{\le} 0$$

$$\Leftrightarrow \frac{-s(s-a)(b-c)^2}{bc(s-b)(s-c)} \overset{?}{\le} 0 \to \text{true} \Rightarrow (*) \text{ is true}$$

$$\therefore \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}} \boxed{\le} \sqrt{\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}}+\sqrt{\frac{\tan\frac{C}{2}}{\tan\frac{B}{2}}} \text{ and finally,}$$

$$w_a.\left(\sqrt{\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}}+\sqrt{\frac{\tan\frac{C}{2}}{\tan\frac{B}{2}}}\right) \le \sqrt{s(s-a)}.\left(\sqrt{\frac{s-c}{s-b}}+\sqrt{\frac{s-b}{s-c}}\right)$$

$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s-b} + \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s-c} = r_b + r_c$$

$$\therefore \sqrt{\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}} + \sqrt{\frac{\tan\frac{C}{2}}{\tan\frac{B}{2}}} \boxed{\le} \frac{r_b + r_c}{w_a} \text{ and so, } max\left\{\frac{2w_a}{h_a}, \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}} + \sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}}\right\} \le$$

$$\sqrt{\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}} + \sqrt{\frac{\tan\frac{C}{2}}{\tan\frac{B}{2}}} \le \frac{r_b + r_c}{w_a} \ \forall \Delta ABC, '' ='' \text{ iff } b = c \ (QED)$$