

ROMANIAN MATHEMATICAL MAGAZINE

If in $\triangle ABC$ $a = \max\{a, b, c\}$ then:

$$\frac{b}{c} + \frac{c}{b} \leq \frac{s^2}{4Rr} - \frac{11}{8}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

We need to show :

$$\frac{b}{c} + \frac{c}{b} \leq \frac{s^2}{4Rr} - \frac{11}{8} \text{ or, } \frac{s^2}{4Rr} - \left(\frac{b}{c} + \frac{c}{b} \right) \geq \frac{11}{8}$$

now, we will show $(a+b+c)^3 - 8a(b^2+c^2) \geq 11abc$
 or, $(a+b+c)^3 - 8a(b^2+c^2) - 11abc \geq 0$
 $a = \max\{a, b, c\}$.

Let $b = a - x, c = a - y$ where $x, y \geq 0$

$$(a+b+c)^3 - 8a(b^2+c^2) - 11abc =$$

$$= (3a-x-y)^3 - 8a((a-x)^2 + (a-y)^2) - 11a(a-x)(a-y) =$$

$$= ax^2 + ay^2 + 7axy - x^3 - 3x^2y - 3xy^2 - y^3 = a((x+y)^2 + 5xy) - (x+y)^3 =$$

$$= (x+y)^2(a-x-y) + 5axy = (x+y)^2(b+c-a) + 5axy \geq 0 \text{ true}$$

(as $b = a - x, c = a - y, x, y \geq 0$ then $a - x - y = b + c - a > 0$ in any triangle)

$$\text{so } (a+b+c)^3 - 8a(b^2+c^2) \geq 11abc$$

$$R.H.S = \frac{s^2}{4Rr} - \left(\frac{b}{c} + \frac{c}{b} \right) = \frac{s^3}{4Rrs} - \frac{b^2+c^2}{bc} = \frac{(a+b+c)^3}{8abc} - \frac{b^2+c^2}{bc} =$$

$$= \frac{(a+b+c)^3 - 8a(b^2+c^2)}{8abc} \geq \frac{11abc}{8abc} = \frac{11}{8}$$

Equality holds for an equilateral triangle.